Queen Mary University of London

LTCC 2010 **Applied Dynamical Systems**

Exercise Sheet to Lectures 5 - 10 **Rainer Klages**

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1. Prove Eq. (1.7) in the lecture notes, that is,

$$\left.\frac{\mathrm{d}F^n(x)}{\mathrm{d}x}\right|_{x=x_0} = F'(x_{n-1})F'(x_{n-2})\ldots F'(x_0) \quad .$$

2. Consider the map defined by the function

$$E(x) = \begin{cases} 2x+1 & , \quad -1 \le x < -1/2 \\ 2x & , \quad -1/2 \le x < 1/2 \\ 2x-1 & , \quad 1/2 \le x \le 1\,. \end{cases}$$

   Draw the graph of this map. Is E ergodic? Prove your answer.

3. Consider the Frobenius-Perron equation

$$\rho_{n+1}(x) = \sum_{x=G(x^i)} \rho_n(x^i)|G'(x^i)|^{-1} =: P\rho_n(x)$$

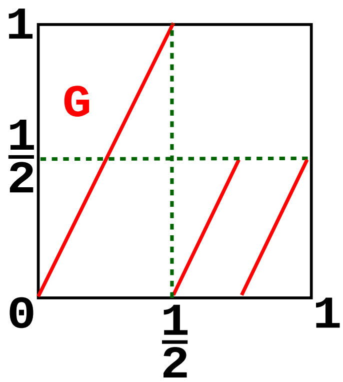
   for a map G defined on the real line, where $\rho_n(x)$ are probability densities at time step $n \in \mathbb{N}_0$ and P defines the Frobenius-Perron operator.

   (a) Show that P is a linear and positive operator.

   (b) Construct P for the map G defined in the figure below and verify that

$$\rho^*(x) = \begin{cases} 4/3 & , \quad 0 \le x < 1/2 \\ 2/3 & , \quad 1/2 \le x \le 1 \end{cases}$$

   is an invariant density of the above Frobenius-Perron equation.



   (c) By assuming that G is ergodic, calculate the Ljapunov exponent λ for this map.

4. Consider the asymmetric tent map

$$S(x) = \begin{cases} ax & , \quad 0 \le x < 1/a \\ b - bx & , \quad 1/a \le x \le 1 \end{cases}$$

with $1/a + 1/b = 1$.

   (a) Calculate the Ljapunov exponent λ for this map.

   (b) Show that for the H-function defined by Eq. (1.27) in the lecture notes it holds $H(\{J_i^2\}) = 2H(\{J_i^1\})$. By assuming that $\forall n \in \mathbb{N}\ H(\{J_i^n\}) = nH(\{J_i^1\})$ calculate the KS-entropy h_{KS}. Compare your result for h_{KS} with the one obtained for λ.

5. Consider the map $H(x) = 5x \mod 1$ on a domain which has "holes" where points escape from the unit interval. Let these escape regions be defined by the two subintervals $(0.2, 04)$ and $(0.6, 0.8)$.

   (a) Sketch the map and the first two steps in the construction of its fractal repeller $\mathcal{R}_H$.

   (b) Calculate the escape rate $\gamma(\mathcal{R}_H)$.

   (c) Calculate the Ljapunov exponent $\lambda(\mathcal{R}_H)$

   (d) Calculate the KS-entropy $h_{KS}(\mathcal{R}_H)$.

   (e) By using these results, verify the escape rate formula for this map.

6. Verify Eq. (2.46) in the lecture notes by using Eqs. (2.44), (2.45).

Model solutions will be available on the course webpage after all lectures are over.