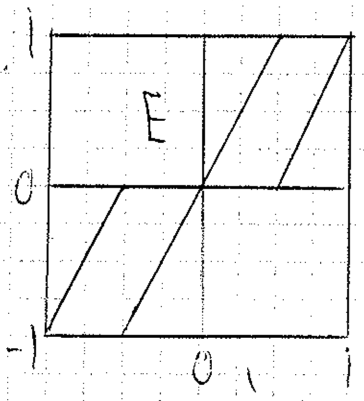
LTCC Course on Applied Dynamical Systems
Solutions to the Exercise Sheet

1. $$(F^n)'(x) \underset{(1.2)}{=} (F(F^{n-1}))'(x) = F'(F^{n-1}(x)) \cdot (F^{n-1})'(x) =$$
$$= F'(F^{n-1}(x)) \cdot F'(F^{n-2}(x)) \cdot (F^{n-2})'(x) = \ldots =$$
$$= F'(F^{n-1}(x)) \cdot F'(F^{n-2}(x)) \cdot \ldots \cdot F'(x) =$$
$$= F'(x_{n-1}) \cdot F'(x_{n-2}) \cdot \ldots \cdot F'(x_0) \quad \text{with } x = x_0$$

2. 

Choose for g the indicator function

$$g : [-1,1] \to \{0,1\},$$
$$g(x) := \begin{cases} 0, & x \in [-1,0) \\ 1, & x \in [0,1] \end{cases}$$

For g it is $\int_{-1}^{1} d\mu^* |g(x)| = \int_{-1}^{0} d\mu^* \underbrace{|g|}_{=0} + \int_{0}^{1} d\mu^* \underbrace{|g|}_{=1} =$
$$= \int_0^1 d\mu^* = \int_0^1 dx\, \rho^*(x) = \frac{1}{2} < \infty \quad (\mu^* \text{ is a probability measure})$$
fulfilled but for $x \in [-1,0) \Rightarrow g(x) = 0$, whereas for
$$x \in [0,1] \Rightarrow g(x) = 1$$
Hence $g(x)$ depends on x and thus $g(x) \neq$ const. in contradiction to def. 4 $\Rightarrow$ E is not ergodic.

3. (a) linearity: $P(\lambda_1 \rho^1 + \lambda_2 \rho^2) = \sum_{x=G(x_i)} |G'(x_i)|^{-1} (\lambda_1 \rho^1 + \lambda_2 \rho^2) =$
$$\underset{\lambda_1,\lambda_2 = \text{const.}}{=} \lambda_1 \sum_{x=G(x_i)} |G'(x_i)|^{-1} \rho^1(x_i) + \lambda_2 \sum_{x=G(x_i)} |G'(x_i)|^{-1} \rho^2(x_i) =$$
$$= \lambda_1 P\rho^1 + \lambda_2 P\rho^2, \quad \lambda_1, \lambda_2 \in \mathbb{R}, \ \rho^1, \rho^2 \text{ prob. densities}$$

positivity: $|G'(x^i)| \geq 0$ (and $|G'(x^i)| = 0$ excluded because it must $\exists$ inverse)

$\rho(x^i) \geq 0$ by definition of prob. density

$\Rightarrow$ right hand side of FP-eqn must be $\geq 0 \Rightarrow \underline{\underline{P\rho(x) \geq 0}}$

(b) from figure:

$$G(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x < \frac{3}{4} \\ 2x - \frac{3}{2}, & \frac{3}{4} \leq x \leq 1 \end{cases}$$

and $G'(x) = 2 \ \forall x \ (x \neq \frac{1}{2}, \frac{3}{4})$

$x^1 = G^{-1}(x) = \frac{x}{2}, \quad 0 \leq x^1 < \frac{1}{2}, \quad 0 \leq x < 1$

$x^2 = G^{-1}(x) = (x+1)/2, \quad \frac{1}{2} \leq x^2 < \frac{3}{4}, \quad 0 \leq x < \frac{1}{2}$

$x^3 = G^{-1}(x) = (x + \frac{3}{2})/2, \quad \frac{3}{4} \leq x^3 \leq 1, \quad 0 \leq x \leq \frac{1}{2}$

$$\Rightarrow \underline{\underline{P\rho(x)}} = \underline{\underline{\frac{1}{2}\rho\left(\frac{x}{2}\right) + \frac{1}{2}\rho\left(\frac{x+1}{2}\right) + \frac{1}{2}\rho\left(\frac{x}{2} + \frac{3}{4}\right)}}$$

for $0 \leq x \leq \frac{1}{2}$: $\underline{\underline{\frac{4}{3}}} = \rho^*(x) = P\rho^*(x) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^2}{3} + \frac{1}{\cancel{2}}\frac{\cancel{2}}{3} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} = \underline{\underline{\frac{4}{3}}}$ ✓

$\frac{1}{2} < x \leq 1$: $\underline{\underline{\frac{2}{3}}} = \rho^*(x) = P\rho^*(x) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^2}{3} = \underline{\underline{\frac{2}{3}}}$ ✓

(c) following Example 4, Eq. (1.19) we obtain

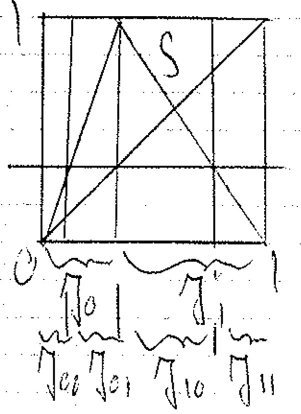
$$\underline{\underline{\lambda}} = \int_0^1 dx\, \rho^*(x) \ln|G'(x)| = \ln 2 \underbrace{\int_0^1 dx\, \rho^*(x)}_{=1 \text{ normalization}} =$$

$$= \underline{\underline{\ln 2}}$$

4. (a) It is easy to see that $\varrho^*(x) = 1$ for this map (otherwise check as in 3. (b)). With this we have

$$\underline{\underline{\lambda}} = \int_0^1 dx\, \varrho^*(x) \ln |S'(x)| = \underline{\underline{\frac{1}{a}\ln a + \frac{1}{b}\ln b}}$$

(b) Let's follow p. 13–15 in the lecture notes:



1. It is convenient to choose as a partition the one generated by backward iteration of the critical point at $x_c = \frac{1}{a}$, see the figure

2. $H(\{J_i^1\}) = \frac{1}{a}\ln a + \frac{1}{b}\ln b$

$$\underline{H(\{J_i^2\})} = \left(\frac{1}{a}\right)^2 \ln a^2 + \frac{1}{ab}\ln(ab) + \left(\frac{1}{b}\right)^2 \ln b^2 + \frac{1}{ba}\ln(ab) =$$

$$= 2\frac{1}{a^2}\ln a + 2\frac{1}{ab}\ln a + 2\frac{1}{ab}\ln b + 2\frac{1}{b^2}\ln b =$$

$$= 2\frac{1}{a}\left(\frac{1}{a}+\frac{1}{b}\right)\ln a + 2\frac{1}{b}\left(\frac{1}{b}+\frac{1}{a}\right)\ln b =$$

$$= \underline{2H(\{J_i^1\})}$$

3. with $\underline{\underline{h(\{J_i^n\})}} = \lim_{n\to\infty} \frac{1}{n} H(\{J_i^n\}) \overset{\text{assumption}}{=} \underline{\underline{H(\{J_i^1\})}}$

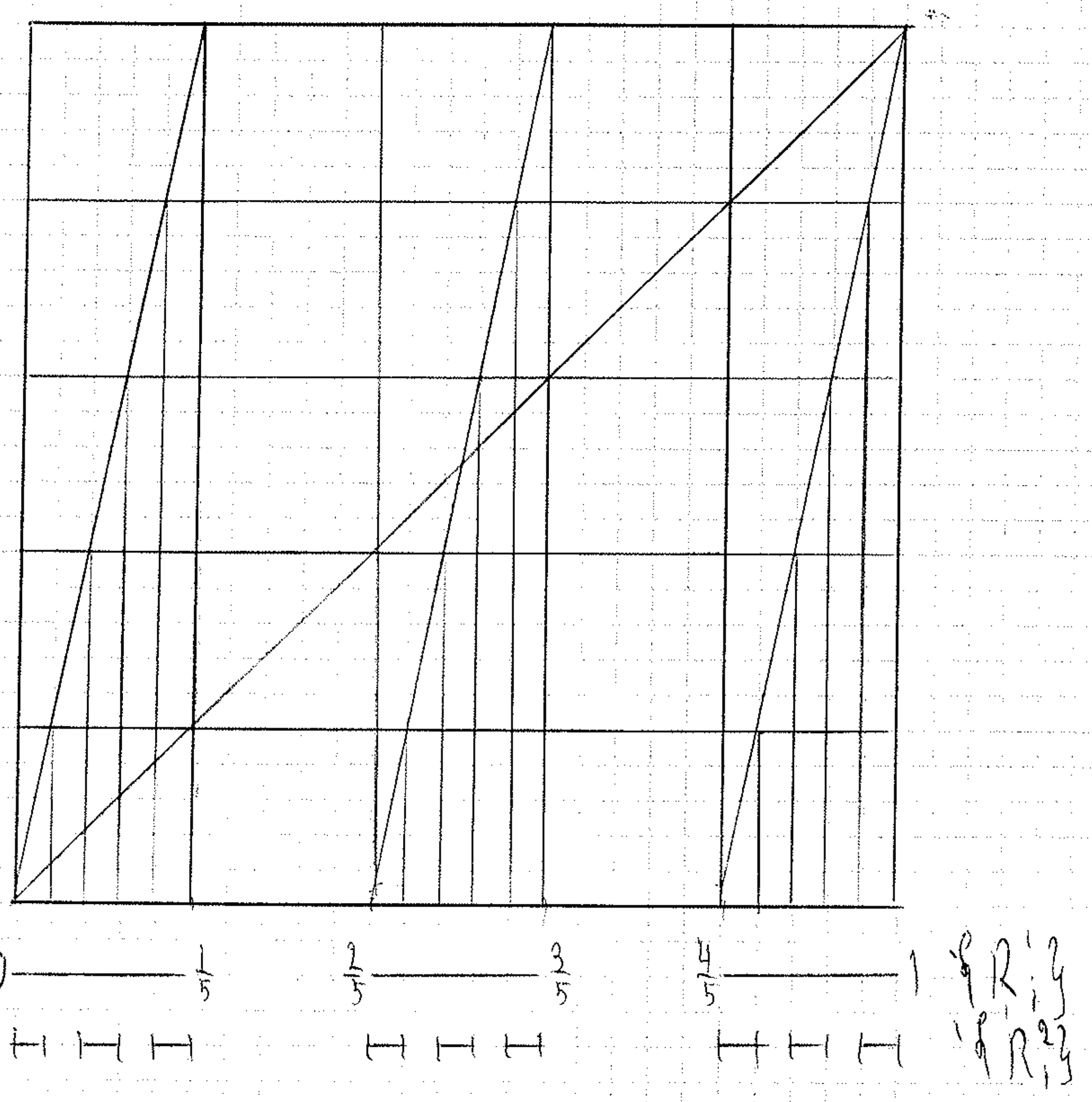
4. The partition chosen above is generating, hence

$$\underline{h_{KS}} = h(\{J_i^n\}) = \underline{\frac{1}{a}\ln a + \frac{1}{b}\ln b} = \lambda,$$

which is Pesin's theorem.

[This exercise is taken from Dorfman's book Ref. [Dor 99], p. 132; see also [Bec 93] p. 153ff]

5. (a)



(b) Since $\rho^*(x)=1$, μ^* just coincides with the Lebesgue measure of the sets $\{R_i^n\}$, i.e., the total length l_n:

$l_0 = 1$; $l_1 = \frac{3}{5}$; $l_2 = \frac{9}{25}$ $\Rightarrow$ $\underline{\underline{l_n = \left(\frac{3}{5}\right)^n}}$,

hence $l_n = e^{-n \ln \frac{5}{3}}$, so $\underline{\underline{\gamma(R_H) = \ln \frac{5}{3}}}$

(c) $\underline{\underline{\lambda(R_H)}} = \int_0^1 d\mu^* \ln|H'(x)| = \underline{\underline{\ln 5}}$

(d) cf. Example 12, p. 20 ff:

$$\mu^*(R_i^1) = \frac{1/5}{3/5} = \frac{1}{3}, \quad \mu^*(R_i^2) = \frac{1/25}{9/25} = \frac{1}{9}$$

$$\Rightarrow \mu^*(R_i^n) = \left(\frac{1}{3}\right)^n,$$

Therefore $H(\{R_i^n\}) = -\sum^{3^n} 3^{-n} \ln 3^{-n} = n \ln 3$

and $h_{KS}(R_H) = \lim_{n\to\infty} \frac{1}{n} H(\{R_i^n\}) = \ln 3$

(e) We thus have the escape rate formula

$$\gamma(R_H) = \ln\frac{5}{3} = \ln 5 - \ln 3 = \lambda(R_H) - h_{KS}(R_H)$$

6. with $\cos x \simeq 1 - \frac{x^2}{2}$, $\ln(1 \pm x) \simeq \pm x$ we have

$$\gamma_{FP} \simeq \ln \frac{4}{2 + 2 - \left(\frac{\pi}{L+1}\right)^2} \simeq \ln \frac{1}{1 - \frac{1}{4}\left(\frac{\pi}{L+1}\right)^2} \simeq$$

$$\simeq -\ln\left(1 - \frac{1}{4}\left(\frac{\pi}{L+1}\right)^2\right) \simeq \left(\frac{1}{4}\left(\frac{\pi}{L+1}\right)^2\right)$$

leads to Eq. (2.46)