MAS424/MTHM021 Introduction to Dynamical Systems

Exercise Sheet 1 **Rainer Klages**

1. *The harmonic oscillator*
   Consider the differential equation
   $$\ddot{x} + \omega^2 x = 0 \; , \; x \in \mathbb{R} \; , \; t \geq 0 \; ,$$
   where $\omega > 0$ is a parameter.

   (a) Solve this differential equation for the initial conditions $x(0) = 0$, $\dot{x}(0) = v_0$.

   (b) Depict your solution graphically by drawing trajectories in the phase space of the system for different values of v_0 (this is called a *phase portrait*).

   (c) Let us assume that 'chaos' is a subset of complicated dynamics. According to the Poincaré-Bendixson theorem, is chaos possible in this dynamical system?

2. *The Poincaré-Bendixson theorem again*

   (a) Is chaos possible in the chain of coupled harmonic oscillators defined by
   $$\begin{aligned} m_1\ddot{x}_1 &= -k_{12}(x_1 - x_2) \\ m_2\ddot{x}_2 &= -k_{12}(x_2 - x_1) - k_{23}(x_2 - x_3) \\ m_3\ddot{x}_3 &= -k_{23}(x_3 - x_2) \quad , \end{aligned}$$
   where $x \in \mathbb{R}$, $t \geq 0$, where m_i , $k_{ij} > 0$, $i, j = 1, 2, 3$ are all parameters?

   (b) Consider the three maps $T(x) = x/2$ ($x \in \mathbb{R}$), $V(x) = 2x$ mod1 and $W(x) = 4x$ for $-0.5 \leq x < 0.5$ with $W(x + 1) = W(x) + 1$ ($x \in \mathbb{R}$). Can they exhibit chaos? Draw graphs of these maps, including cobweb plots for initial conditions of your choice, to illustrate your answers.

   (c) Let
   $$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= y_n + k \sin x_{n+1} \; , \; n \in \mathbb{N} \; , \; (x_n, y_n) \in \mathbb{R}^2 \; , \end{aligned}$$
   be the (standard) map, where $k > 0$ is a parameter. Is this map invertible? Justify your answer. According to the Poincaré-Bendixson theorem, is chaos possible in this dynamical system?

   (d) Let us consider a vector field on the unit square in the plane with periodic boundary conditions, where each vector has slope $q \in \mathbb{R}$. According to the Poincaré-Bendixson theorem, is chaos possible? Let us consider this vector field on a torus, i.e., by gluing together the left and right edges of the square, and likewise the top and bottom ones. Is chaos possible? *(hint: in case of trouble with this question the book by Alligood et al. might help)*

3. *Cobweb plots and periodic orbits*

   (a) Consider the map $C : \mathbb{R} \to \mathbb{R}$, $x_{n+1} = C(x_n)$, $n \in \mathbb{N}$ defined by the function $C(x) = -x^2 + x + 2$, $x \in \mathbb{R}$. Calculate the set $Per_2(C)$ of all period 2 points for this map. Draw the graph of $C(x)$ and mark the positions of all period 2 points. Include cobweb plots for all period two orbits and illustrate the stability of the fixed points by cobweb plots.

   (b) Draw a cobweb plot for a one-dimensional map of your choice showing a prime period three orbit and an eventually periodic orbit.

4. *Rotation on the circle*

   Let S^1 be the unit circle in the plane. Let denote a point in S^1 by its angle θ such that a point on the circle is determined by any angle of the form $\theta + 2k\pi$ for an integer k. Now let $R_\lambda(\theta) = \theta + 2\pi\lambda$ be a rotation on the circle. Show that if λ is rational then every $\theta \in S^1$ is a periodic point. Prove by contradiction that there are no periodic points if λ is irrational.

Model solutions will be on the course webpage starting from Thursday, October 25, 2007.