LOWER BOUNDS FOR THE RUELLE SPECTRUM OF ANALYTIC EXPANDING CIRCLE MAPS

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Abstract. We prove that there exists a dense set of analytic expanding maps of the circle for which the Ruelle eigenvalues enjoy exponential lower bounds. The proof combines potential theoretic techniques and explicit calculations for the spectrum of expanding Blaschke products.

1. Introduction and statement

One of the basic problems of smooth ergodic theory is to investigate the asymptotic behaviour of mixing systems. In particular, there is interest in precise quantitative results on the rate of decay of correlations for smooth observables. In his seminal paper [17], Ruelle showed that the long time asymptotic behaviour of *analytic* hyperbolic systems can be understood in terms of the *Ruelle spectrum*, that is, the spectrum of certain operators, known as transfer operators in this context, acting on suitable Banach spaces of holomorphic functions. Surprisingly, even for the simplest systems like analytic expanding maps of the circle, very few quantitative results on the Ruelle spectrum are known so far.

Let us be more precise. With $\mathbb{T}$ denoting the unit circle in the complex plane $\mathbb{C}$, a map $\tau : \mathbb{T} \to \mathbb{T}$ is said to be *analytic expanding* if τ has a holomorphic extension to a neighbourhood of $\mathbb{T}$ and we have

$$\inf_{z\in\mathbb{T}} |\tau'(z)| > 1\,.$$

The Ruelle spectrum is commonly defined as the spectrum of the Perron-Frobenius or transfer operator (on a suitable space of holomorphic functions) given locally by

$$(\mathcal{L}_\tau f)(z) := \omega(\tau)\sum_{k=1}^{d} \phi_k'(z) f(\phi_k(z))\,, \tag{1}$$

where ϕ_k denotes the k-th local inverse branch of the covering map $\tau : \mathbb{T} \to \mathbb{T}$, and

$$\omega(\tau) = \begin{cases} +1 & \text{if } \tau \text{ is orientation preserving;} \\ -1 & \text{if } \tau \text{ is orientation reversing.} \end{cases} \tag{2}$$

As first demonstrated by Ruelle in [17], this transfer operator has a discrete spectrum of eigenvalues which has an intrinsic dynamical meaning and does not depend on the choice of function space. Let $(\lambda_n(\mathcal{L}_\tau))_{n\in\mathbb{N}}$ denote this eigenvalue sequence, counting algebraic multiplicities and ordered by decreasing modulus, so that

$$1 = \lambda_1(\mathcal{L}_\tau) > |\lambda_2(\mathcal{L}_\tau)| \geq |\lambda_3(\mathcal{L}_\tau)| \geq \cdots \geq |\lambda_n(\mathcal{L}_\tau)| \geq \cdots \geq 0\,.$$

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To state the main result, given $0 < r < 1 < R$, we denote by $A_{r,R}$ the complex annulus defined by

$$A_{r,R} = \{ z \in \mathbb{C} : r < |z| < R \} \supset \mathbb{T} .$$

In this paper we prove the following facts.

Theorem 1.1. *Let τ be an analytic expanding map of the circle as defined above. Then the following holds.*

(1) *There exist constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ we have*
$$|\lambda_n(\mathcal{L}_\tau)| \leq c_1 \exp(-c_2 n) .$$

(2) *Assume that τ is holomorphic on $A_{r,R}$ for some $0 < r < 1 < R$. Then there exist $r < r_1 < 1 < R_1 < R$ such that for all $\eta > 0$, one can find an expanding circle map τ_η holomorphic on A_{r_1,R_1} with*
$$\sup_{z \in A_{r_1,R_1}} |\tau(z) - \tau_\eta(z)| \leq \eta,$$
and such that for all $\epsilon > 0$, we have
$$\limsup_{n\to\infty} |\lambda_n(\mathcal{L}_{\tau_\eta})| \exp(n^{1+\epsilon}) > 0 .$$

The first statement is a standard fact and follows from the original paper of Ruelle: the sequence of eigenvalues enjoys an exponential upper bound. The second statement shows that for a dense set of analytic circle maps, the Ruelle spectrum is infinite and has a purely exponential decay: the upper bound is optimal. Alternatively, if one sets for $r > 0$

$$N_\tau(r) := \#\{ n \in \mathbb{N} : |\lambda_n(\mathcal{L}_\tau)| \geq r \} ,$$

then the above theorem says that for all analytic expanding maps τ we have, as $r \to 0$,

$$N_\tau(r) = O(|\log(r)|) ,$$

while for a dense set of analytic maps τ we have, for all $\epsilon > 0$,

$$N_\tau(r) = \Omega\left(|\log(r)|^{1-\epsilon}\right) ;$$

here, the "omega" notation $f(r) = \Omega(g(r))$ means that there does *not* exist $C > 0$ such that for all $r > 0$ small we have $f(r) \leq Cg(r)$.

Notice that this statement cannot hold for all analytic expanding circle maps. Indeed, the popular expanding maps $z \mapsto z^d$ have a trivial Ruelle spectrum $\{0, 1\}$, see, for example, [5].

The paper is organised as follows. In the next section, we define a general class of holomorphic maps on annuli that slightly generalises the class of analytic expanding maps of the circle. We show that it is possible to define a Hilbert space of hyperfunctions on which the composition (Koopman) operator has a discrete spectrum of eigenvalues, together with an exponential *a priori* upper bound. In the case of circle maps, this discrete spectrum turns out to be the same as the Ruelle spectrum, by a standard duality argument. In the following Section 3 we revisit the main result from [5] which gives an explicit expression for the spectra of transfer operators arising from Blaschke products. After this, we shall derive a similar expression for the spectra of transfer operators arising from anti-Blaschke products (defined as the reciprocals of Blaschke products). Both these results will be used in a critical way later on in the proof of our main result. In Section 4, we recall the necessary potential theoretic background which is the core of the main proof. In Section 5 we prove a key lemma on deformations of circle maps in the space of annulus maps. Finally, in the last section, we gather all the previous facts to give a proof of the main theorem.

We hope that the ideas and techniques used here in a one-dimensional setup can serve as a blueprint for future work related to the Ruelle spectrum of Anosov maps and flows. This will be pursued elsewhere.

2. Holomorphically expansive maps of the annulus and upper spectral bounds for their transfer operators

In this section we first define a class of holomorphic maps on an annulus, termed 'holomorphically expansive', which mildly generalises the class of analytic expanding circle maps. We then introduce Hardy-Hilbert spaces over disks and annuli and show that on these spaces composition operators given by holomorphically expansive maps have exponentially decaying eigenvalues. For the connection with the Ruelle eigenvalue sequence, we recall a useful representation for the dual of the transfer operator arising from analytic expanding circle maps as a composition operator described in [5]. Similar representations for transfer operators associated with certain rational maps have been given in [9], where explicit expressions for the corresponding Fredholm determinants can also be found.

We start by fixing notation. For $r > 0$ we use

$$\mathbb{T}_r = \{z \in \mathbb{C} : |z| = r\},$$

to denote circles centred at 0, and

$$D_r = \{z \in \mathbb{C} : |z| < r\}, \quad D_r^\infty = \{z \in \hat{\mathbb{C}} : |z| > r\}, \quad \mathbb{D} = D_1$$

to denote disks centred at 0 and ∞.

Definition 2.1. Let $A_{r,R}$ be an annulus and let $\tau : A_{r,R} \to \mathbb{C}$. We shall call τ *holomorphically expansive on* $A_{r,R}$, or simply *holomorphically expansive* if the annulus is understood, if τ is holomorphic on the closure of $A_{r,R}$ and we have

$$\tau(A_{r,R}) \supset \mathrm{cl}(A_{r,R}).$$

Here, $\mathrm{cl}(A)$ is the closure of a subset A of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Remark 2.2. Suppose that τ is holomorphic on the closed annulus $\mathrm{cl}(A_{r,R})$. Since any non-constant holomorphic map is open, we must have $\partial\tau(A_{r,R}) = \tau(\partial A_{r,R})$ and it follows that τ is holomorphically expansive on $A_{r,R}$ if and only if either of the following two alternatives hold:

(A1) $\tau(\mathbb{T}_r) \subset D_r$ and $\tau(\mathbb{T}_R) \subset D_R^\infty$;

(A2) $\tau(\mathbb{T}_r) \subset D_R^\infty$ and $\tau(\mathbb{T}_R) \subset D_r$.

If (A1) holds we shall call τ *orientation preserving*, while if (A2) holds we shall call τ *orientation reversing*.

It is not difficult to see that every analytic expanding circle map is holomorphically expansive on all sufficiently small annuli containing the unit circle (see, for example, [19, Lemma 2.2]; this is also a special case of Lemma 5.2, to be proved later), but not the other way round. As we shall see shortly, with every holomorphically expansive τ it is possible to associate an operator with discrete spectrum, which, in case τ leaves the unit circle invariant, coincides with the Ruelle spectrum of τ, that is, the spectrum of $\mathcal{L}_\tau$.

In order to make this connection more precise, we first need to introduce appropriate spaces of holomorphic functions on which these operators act. The positively oriented boundary of a disk or an annulus will be denoted by ∂_+. For U an open subset of $\hat{\mathbb{C}}$ we write $\mathrm{Hol}(U)$ to denote the space of holomorphic functions on U.

Definition 2.3. For $\rho > 0$ and $f : \mathbb{T}_\rho \to \mathbb{C}$ write

$$M_\rho(f) = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^2 \, d\theta \, .$$

Then

$$\begin{aligned} H^2(D_r) &= \{ f \in \mathrm{Hol}(D_r) \, : \, \sup_{\rho<r} M_\rho(f) < \infty \} \\ H^2(A_{r,R}) &= \{ f \in \mathrm{Hol}(A_{r,R}) \, : \, \sup_{\rho>r} M_\rho(f) + \sup_{\rho<R} M_\rho(f) < \infty \} \\ H^2(D_R^\infty) &= \{ f \in \mathrm{Hol}(D_r^\infty) \, : \, \sup_{\rho>R} M_\rho(f) < \infty \} \end{aligned}$$

are called the *Hardy-Hilbert spaces* on D_r, $A_{r,R}$, and D_R^∞, respectively.

The subspace of $H^2(D_R^\infty)$ consisting of functions vanishing at infinity will be denoted by $H_0^2(D_R^\infty)$.

The classic text [7] gives a comprehensive account of Hardy spaces over general domains. Hardy spaces on the unit disk are discussed in considerable detail in [16, Chapter 17]), while a good reference for Hardy spaces on annuli is [18].

We briefly mention a number of results which will be useful in what follows. Any function in $H^2(U)$, where U is a disk or an annulus, can be extended to the boundary in the following sense. For any $f \in H^2(D_r)$ there is $f^* \in L^2(\mathbb{T}_r) = L^2(\mathbb{T}_r, d\theta/2\pi)$ the usual Hilbert space of square-integrable functions with respect to normalized one-dimensional Lebesgue measure on $\mathbb{T}_r$, such that

$$\lim_{\rho\uparrow r} f(\rho e^{i\theta}) = f^*(re^{i\theta}) \quad \text{for a.e. } \theta,$$

and analogously for $f \in H^2(D_R^\infty)$. Similarly, for $f \in H^2(A_{r,R})$ there are $f_1^* \in L^2(\mathbb{T}_r)$ and $f_2^* \in L^2(\mathbb{T}_R)$, with

$$\lim_{\rho\downarrow r} f(\rho e^{i\theta}) = f_1^*(re^{i\theta}) \text{ and } \lim_{\rho\uparrow R} f(\rho e^{i\theta}) = f_2^*(Re^{i\theta}) \text{ for a.e. } \theta \, .$$

The above terminology is justified since the spaces $H^2(U)$ turn out to be Hilbert spaces with inner products

$$(f,g)_{H^2(D_r)} = \frac{1}{2\pi}\int_0^{2\pi} f^*(re^{i\theta})\overline{g^*(re^{i\theta})} \, d\theta$$

and

$$(f,g)_{H^2(A_{r,R})} = \frac{1}{2\pi}\int_0^{2\pi} f_1^*(Re^{i\theta})\overline{g_1^*(Re^{i\theta})} \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} f_2^*(re^{i\theta})\overline{g_2^*(re^{i\theta})} \, d\theta \, .$$

Similarly, for $H^2(D_R^\infty)$.

Remark 2.4. In the following we shall write $f(z)$ instead of $f^*(z)$ for z on the boundary of the domain, if this does not lead to confusion.

Remark 2.5. For $\rho > 0$ and $n \in \mathbb{Z}$ let

$$e_n^{(\rho)}(z) = \frac{z^n}{\rho^n} \, .$$

It is not difficult to see that $\{ e_n^{(\rho)} \, : \, n \in \mathbb{N}_0 \}$ is an orthonormal basis for $H^2(D_\rho)$ and that $\{ e_{-n}^{(\rho)} \, : \, n \in \mathbb{N} \}$ is an orthonormal basis for $H_0^2(D_\rho^\infty)$.

For later use, we note the following simple consequence of the above remark.

Lemma 2.6. *For any $f \in H^2(D_r)$ and any $z \in D_r$ we have*

$$|f(z)| \leq \frac{r}{\sqrt{r^2 - |z|^2}} \|f\|_{H^2(D_r)} . \tag{3}$$

Similarly, for any $f \in H^2_0(D_R^\infty)$ and any $z \in D_R^\infty$ we have

$$|f(z)| \leq \frac{R}{\sqrt{|z|^2 - R^2}} \|f\|_{H^2_0(D_R^\infty)} . \tag{4}$$

Proof. We shall only prove (4). The proof of (3) is similar. Let $f \in H^2_0(D_R^\infty)$. Then f has an orthonormal expansion of the form

$$f = \sum_{n=1}^{\infty} f_n e_{-n}^{(R)} .$$

The Cauchy-Schwarz inequality now implies that for $z \in D_R^\infty$ we have

$$|f(z)|^2 \leq \sum_{n=1}^{\infty} |f_n|^2 \sum_{n=1}^{\infty} \frac{|R|^{2n}}{|z|^{2n}} = \|f\|^2_{H^2_0(D_R^\infty)} \frac{R^2}{|z|^2 - R^2}$$

and the assertion follows. □

We now recall a number of facts from [5] which will be crucial for what is to follow. We start with the simple observation that if τ is an analytic expanding circle map which is holomorphically expansive on an annulus $A_{r,R}$, then the corresponding transfer operator $\mathcal{L}_\tau$ is an endomorphism of $H^2(A_{r,R})$. In fact, as we shall see later, $\mathcal{L}_\tau$ has much stronger functional analytic properties on $H^2(A_{r,R})$.

The next fact is concerned with the strong dual $H^2(A_{r,R})'$ of $H^2(A_{r,R})$, that is, the space of continuous linear functionals on $H^2(A_{r,R})$ equipped with the topology of uniform convergence on the unit ball. It turns out that it can be represented in terms of the topological direct sum $H^2(D_r) \oplus H^2_0(D_R^\infty)$, equipped with the norm $\|(h_1, h_2)\|^2 = \|h_1\|^2_{H^2(D_r)} + \|h_2\|^2_{H^2_0(D_R^\infty)}$, turning it into a Hilbert space.

Proposition 2.7. *The dual space $H^2(A_{r,R})'$ is isomorphic to $H^2(D_r) \oplus H^2_0(D_R^\infty)$ with the isomorphism given by*

$$H^2(D_r) \oplus H^2_0(D_R^\infty) \to H^2(A_{r,R})'$$
$$(h_1, h_2) \mapsto l ,$$

where

$$l(f) = \frac{1}{2\pi i} \int_{\partial_+ D_r} f(z) h_1(z)\, dz + \frac{1}{2\pi i} \int_{\partial_+ D_R} f(z) h_2(z)\, dz \quad (f \in H^2(A_{r,R})) . \tag{5}$$

Proof. See [5] for a short proof. See also [15, Proposition 3], where similar representations for the duals of Hardy spaces over multiply connected regions are provided. □

Next we note that, given a circle map τ, we can associate with it the corresponding *composition operator* C_τ defined for $f : \mathbb{T} \to \mathbb{C}$ by

$$C_\tau f = f \circ \tau ,$$

which, in the context of dynamical systems, is also known as *Koopman operator.* Moreover, if τ is an analytic expanding circle map which is holomorphically expansive on an annulus $A_{r,R}$, then $\mathcal{L}'_\tau$, the Banach space-adjoint[1] of the corresponding transfer operator $\mathcal{L}_\tau$, can be represented as a compression of C_τ to $H^2(D_r) \oplus H^2_0(D^\infty_R)$.

In order to make this connection more precise, we need to introduce certain projection operators on $L^2(\mathbb{T}_\rho)$. For any $f \in L^2(\mathbb{T}_\rho)$ we can write $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$, so that $f = f_+ + f_-$ with $f_+(z) = \sum_{n=0}^\infty f_n z^n$ and $f_-(z) = \sum_{n=1}^\infty f_{-n} z^{-n}$. Since $\|f\|^2_{L^2(\mathbb{T}_\rho)} = \sum_{n=-\infty}^\infty |f_n|^2 \rho^{2n} < \infty$, the functions f_+ and f_- can be viewed as functions in $H^2(D_\rho)$ and $H^2_0(D^\infty_\rho)$, respectively. Thus, we can define two projection operators $\Pi^{(\rho)}_+ \colon L^2(\mathbb{T}_\rho) \to H^2(D_\rho)$ and $\Pi^{(\rho)}_- \colon L^2(\mathbb{T}_\rho) \to H^2_0(D^\infty_\rho)$ by setting

$$\Pi^{(\rho)}_+ f = f_+ \quad \text{and} \quad \Pi^{(\rho)}_- f = f_- \,, \tag{6}$$

which are easily seen to be bounded. The representation alluded to above can now be stated as follows.

Proposition 2.8. *Let τ be an analytic expanding circle map which is holomorphically expansive on $A_{r,R}$ and let $\mathcal{L}_\tau \colon H^2(A_{r,R}) \to H^2(A_{r,R})$ be the corresponding transfer operator. Then, using the isomorphism given in Proposition 2.7, the adjoint $\mathcal{L}'_\tau$ can be represented by the operator*

$$\mathcal{L}^\dagger_\tau \colon H^2(D_r) \oplus H^2_0(D^\infty_R) \to H^2(D_r) \oplus H^2_0(D^\infty_R) \,,$$

where

$$\mathcal{L}^\dagger_\tau = \begin{pmatrix} \Pi^{(r)}_+ C_\tau & \Pi^{(R)}_+ C_\tau \\ \Pi^{(r)}_- C_\tau & \Pi^{(R)}_- C_\tau \end{pmatrix} \tag{7}$$

if τ is orientation preserving and

$$\mathcal{L}^\dagger_\tau := - \begin{pmatrix} \Pi^{(R)}_+ C_\tau & \Pi^{(r)}_+ C_\tau \\ \Pi^{(R)}_- C_\tau & \Pi^{(r)}_- C_\tau \end{pmatrix} \tag{8}$$

if τ is orientation reversing.

Proof. See [5]. □

We now make an important observation: the operator $\mathcal{L}^\dagger_\tau$ makes sense even if τ does not preserve the unit circle, but is merely holomorphically expansive. In fact, as we shall see shortly, the operator $\mathcal{L}^\dagger_\tau$ has strong spectral properties for any holomorphically expansive τ.

These spectral properties are conveniently described in terms of the theory of exponential classes developed in [1], which we now briefly outline. Recall that if $L : H \to H$ is a compact operator on a Hilbert space H, we use $(\lambda_n(L))_{n\in\mathbb{N}}$ to denote its eigenvalue sequence, counting algebraic multiplicities and ordered by decreasing modulus so that

$$|\lambda_1(L)| \geq |\lambda_2(L)| \geq \cdots$$

If L has only finitely man non-zero eigenvalues, we set $\lambda_n(L) = 0$ for $n > N$, where N denotes the number of non-zero eigenvalues of L. Furthermore, for $L : H_1 \to H_2$ a compact operator between Hilbert spaces H_1 and H_2 and $n \in \mathbb{N}$, the *n-th singular value* of L is given by

$$s_n(L) = \sqrt{\lambda_n(L^*L)} \quad (n \in \mathbb{N}) \,,$$

where L^* denotes the Hilbert space adjoint of L.

[1] Recall that this means that $\mathcal{L}'_\tau : H^2(A_{r,R})' \to H^2(A_{r,R})'$ is given by $(\mathcal{L}'_\tau l)(f) = l(\mathcal{L}_\tau f)$ for all $l \in H^2(A_{r,R})'$ and $f \in H^2(A_{r,R})$.

Definition 2.9. If a compact operator L between Hilbert spaces satisfies

$$s_n(L) \leq c_1 \exp(-c_2 n) \quad (\forall n \in \mathbb{N})$$

for some constants $c_1, c_2 > 0$ we say that L *is of exponential class*. The collection of all compact operators of exponential class will be denoted by $\mathcal{E}$.

Standard examples of operators of exponential class are embeddings between Hardy spaces, as the following lemma shows.

Lemma 2.10. *Let $0 < r' < r$ and let $J : H^2(D_r) \to H^2(D_{r'})$ denote the canonical embedding given by $(Jf)(z) = f(z)$ for $f \in H^2(D_r)$ and $z \in D_{r'}$. Then J is of exponential class.*

Proof. In order to see this note that

$$(J^* J e_n^{(r)}, e_m^{(r)})_{H^2(D_r)} = (J e_n^{(r)}, J e_m^{(r)})_{H^2(D_{r'})} = \delta_{nm} \left(\frac{r'}{r}\right)^{2n}$$

where $(e_n^{(r)})_{n \in N_0}$ denotes the orthonormal basis of $H^2(D_r)$ given in Remark 2.5. Thus

$$s_n(J) = \left(\frac{r'}{r}\right)^{n-1},$$

and it follows that $J \in \mathcal{E}$. □

Remark 2.11. A similar argument shows that for $0 < R < R'$ the canonical embedding $J : H_0^2(D_R) \to H_0^2(D_{R'})$ is of exponential class.

Other examples of naturally occurring operators of exponential class can be found in [2, 3, 4]. In fact, as we shall see shortly, the operator $\mathcal{L}_\tau^\dagger$ is of exponential class for every holomorphically expansive τ, and, moreover, its eigenvalue sequence decays at an exponential rate. The proof of these results relies on the following properties of $\mathcal{E}$.

Proposition 2.12. *The exponential class $\mathcal{E}$ is a two-sided operator ideal, that is, the following two properties hold:*

1. *$L_1, L_2 \in \mathcal{E}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ imply $\alpha_1 L_1 + \alpha_2 L_2 \in \mathcal{E}$, whenever this linear combination is defined;*
2. *if L_1, L_3 are bounded operators and $L_2 \in \mathcal{E}$ then $L_1 L_2 L_3 \in \mathcal{E}$, whenever this product is defined.*

Moreover, if $L \in \mathcal{E}$ is an endomorphism then

$$|\lambda_n(L)| \leq c_1 \exp(-c_2 n) \quad (\forall n \in \mathbb{N})$$

for some constants $c_1, c_2 > 0$.

Proof. See [1, Propositions 2.8 and 2.10] □

We shall now show that the entries of the matrix defining $\mathcal{L}_\tau^\dagger$ in (7) and (8) are well-defined and are each of exponential class for any holomorphically expansive τ.

Proposition 2.13. *Let τ be holomorphically expansive on $A_{r,R}$.*

1. *If τ is orientation preserving then*

$$C_\tau(H^2(D_r)) \subset L^2(\mathbb{T}_r) \text{ and } C_\tau(H^2(D_R^\infty)) \subset L^2(\mathbb{T}_R)\,.$$

(2) *If τ is orientation reversing then*

$$C_\tau(H^2(D_r)) \subset L^2(\mathbb{T}_R) \text{ and } C_\tau(H^2(D_R^\infty)) \subset L^2(\mathbb{T}_r)\,.$$

Moreover in both cases, the restrictions $C_\tau|H^2(D_r)$ and $C_\tau|H^2(D_R^\infty)$ are of exponential class.

Proof. We shall only prove case (2); the other one is similar. We start by observing that, since τ is orientation reversing, we can choose $0 < r' < r$ such that

$$\tau(\mathbb{T}_R) \subset D_{r'} \subset D_r\,.$$

We shall now show that C_τ maps $H^2(D_{r'})$ continuously to $L^2(\mathbb{T}_R)$. In order to see this note that $\tau(\mathbb{T}_R)$ is a compact subset of $D_{r'}$, so

$$C := \sup_{z\in\mathbb{T}_R} \frac{r'}{\sqrt{(r')^2 - |\tau(z)|^2}} < \infty\,.$$

Thus, using Lemma 2.6, we have for any $f \in H^2(D_{r'})$

$$\sup_{z\in\mathbb{T}_R} |f(\tau(z))| \leq C\,\|f\|_{H^2(D_{r'})}\,,$$

and so

$$\|C_\tau f\|_{L^2(\mathbb{T}_R)} \leq C\,\|f\|_{H^2(D_{r'})} \quad (\forall f \in H^2(D_{r'}))\,. \tag{9}$$

We now observe that $C_\tau|H^2(D_r)$ admits a factorisation of the form

$$C_\tau|H^2(D_r) = C_\tau|H^2(D_{r'})J\,,$$

where J denotes the canonical embedding of $H^2(D_r)$ in $H^2(D_{r'})$. Thus, since $C_\tau : H^2(D_{r'}) \to L^2(\mathbb{T}_R)$ is continuous by (9) it follows that C_τ maps $H^2(D_r)$ continuously to $L^2(\mathbb{T}_R)$. Moreover, since J is of exponential class by Lemma 2.10, Proposition 2.12 now implies that $C_\tau|H^2(D_r)$ is of exponential class as well.

A similar argument using Remark 2.11 instead of Lemma 2.10 shows that C_τ maps $H^2(D_R^\infty)$ continuously to $L^2(\mathbb{T}_r)$ and that $C_\tau|H^2(D_R^\infty)$ is of exponential class. □

Corollary 2.14. *If τ is holomorphically expansive then $\mathcal{L}_\tau^\dagger$ is of exponential class and, in particular, its eigenvalue sequence decays exponentially.*

Proof. Follows from Propositions 2.12 and 2.13 □

Remark 2.15. In the corollary above and in the following, we shall always tacitly assume that if τ is holomorphically expansive on $A_{r,R}$, then $\mathcal{L}_\tau^\dagger$ will be considered as an operator from $H^2(D_r) \oplus H_0^2(D_R^\infty)$ to $H^2(D_r) \oplus H_0^2(D_R^\infty)$.

We finish this section with a result that will allow us to calculate the eigenvalue sequence of a particular class of analytic expansive circle maps.

Proposition 2.16. *Let τ be holomorphically expansive on $A_{r,R}$. Then $\mathcal{L}_\tau^\dagger$ is trace class and its trace is given by*

$$\mathrm{Tr}(\mathcal{L}_\tau^\dagger) = \frac{\omega(\tau)}{2\pi i}\int_{\partial_+ A_{r,R}} \frac{1}{\tau(z) - z}\,dz\,.$$

Proof. Clearly, $\mathcal{L}_\tau^\dagger$ is trace class, since by Corollary 2.14 it is of exponential class. In order to calculate its trace we observe that if $\rho \in [r, R]$ and $n \in \mathbb{Z}$ we have

$$(C_\tau e_n^{(\rho)}, e_n^{(\rho)})_{L^2(\mathbb{T}_\rho)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\tau(\rho e^{i\theta})^n}{\rho^n} e^{-in\theta}\,d\theta = \frac{1}{2\pi i}\int_{\partial_+ D_\rho} \frac{\tau(z)^n}{z^{n+1}}\,dz\,. \tag{10}$$

Suppose now that τ is orientation preserving. Then, using the cyclicity of the trace, we have

$$\operatorname{Tr}(\mathcal{L}_\tau^\dagger) = \operatorname{Tr}(\Pi_+^{(r)} C_\tau) + \operatorname{Tr}(\Pi_-^{(R)} C_\tau) = \operatorname{Tr}(C_\tau \Pi_+^{(r)}) + \operatorname{Tr}(C_\tau \Pi_-^{(R)}) \,.$$

But by (10)

$$\operatorname{Tr}(C_\tau \Pi_+^{(r)}) = \sum_{n=-\infty}^{\infty} (C_\tau \Pi_+^{(r)} e_n^{(r)}, e_n^{(r)})_{L^2(\mathbb{T}_r)} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial_+ D_r} \frac{\tau(z)^n}{z^{n+1}}\, dz = \frac{1}{2\pi i} \int_{\partial_+ D_r} \frac{1}{z - \tau(z)}\, dz \,,$$

and

$$\operatorname{Tr}(C_\tau \Pi_-^{(R)}) = \sum_{n=-\infty}^{\infty} (C_\tau \Pi_-^{(R)} e_n^{(R)}, e_n^{(R)})_{L^2(\mathbb{T}_R)} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{\partial_+ D_R} \frac{z^{n-1}}{\tau(z)^n}\, dz = \frac{1}{2\pi i} \int_{\partial_+ D_R^\infty} \frac{1}{z - \tau(z)}\, dz \,,$$

and the assertion follows by observing that $\partial_+ A_{r,R} = \partial_+ D_r^\infty \cup \partial_+ D_R$. The proof for orientation reversing τ is similar. □

3. Blaschke and anti-Blaschke products

In this section we shall consider a particular class of analytic circle maps, for which the eigenvalue sequence of the associated transfer operators can be calculated exactly.

Definition 3.1. For $d \in \mathbb{N}$ let $a = (\alpha, a_1, \ldots, a_d)$ be a $(d+1)$-tuple of complex numbers with $\alpha \in \mathbb{T}$ and $a_1, \ldots, a_d \in \mathbb{D}$. Then

$$B_a(z) = \alpha \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}$$

is called a *Blaschke product of degree d* or a *finite Blaschke product.*

We shall now collect a number of facts about Blaschke products.

Proposition 3.2. *Let B_a be a finite Blaschke product. Then the following holds.*

1. *B_a is meromorphic on $\hat{\mathbb{C}}$ and holomorphic on* $\operatorname{cl}(\mathbb{D})$.
2. *B_a leaves both $\mathbb{T}$ and $\mathbb{D}$ invariant.*
3. *We have $B_a(z^{-1}) = B_{\overline{a}}(z)^{-1}$, where $\overline{a} = (\overline{\alpha}, \overline{a_1}, \ldots, \overline{a_d})$.*
4. *B_a is analytic expanding if and only if it is holomorphically expansive.*
5. *If $\sum_{j=1}^{d} (1 - |a_j|)/(1 + |a_j|) > 1$ then B_a is holomorphically expansive.*
6. *If B_a is holomorphically expansive, then B_a has a unique fixed point z_0 in $\mathbb{D}$ and the corresponding multiplier $B_a'(z_0)$ belongs to $\mathbb{D}$.*

Proof. Part (1) is clear, while (2) and (3) follow from a short calculation. For (4) see [20, Theorem 1] and for (5) see [10, Corollary to Proposition 1]. Finally, (6) follows by combining [13, Proposition 2.1] and [20, Theorem 1]. □

Part (1) and (2) of the above proposition show that a finite Blaschke product yields an analytic circle map. Curiously enough, any analytic circle map which is also holomorphic on $\mathrm{cl}(\mathbb{D})$ is necessarily a finite Blaschke product (see, for example, [6, Exercise 6.12]). In particular, the composition of two finite Blaschke products is again a finite Blaschke product.

It turns out that for expanding circle maps arising from Blaschke products a complete determination of the spectra of the associated transfer operators is possible. For certain Blaschke products of degree 2 this is shown in [19] relying on a block-diagonal matrix representation of the transfer operator. The general case is discussed in [5] using the spectral theory of composition operators with holomorphic symbols. Below we rederive this result by yet another method, exploiting the fact that the trace of $\mathcal{L}_{B_a}^{\dagger}$ is easily calculated whenever B_a is a holomorphically expansive Blaschke product.

Lemma 3.3. *Let B_a be a finite Blaschke product which is holomorphically expansive on $A_{r,R}$. Then*

$$\mathrm{Tr}(\mathcal{L}_{B_a}^{\dagger}) = 1 + \frac{\mu}{1-\mu} + \frac{\overline{\mu}}{1-\overline{\mu}}\,,$$

where μ is the multiplier of the fixed point of B_a in $\mathbb{D}$.

Proof. Let z_0 denote the fixed point of B_a in $\mathbb{D}$. Since B_a is holomorphically expansive on $A_{r,R}$ we must have $z_0 \in D_r$. Thus

$$\frac{1}{2\pi i}\int_{\partial_+ D_r} \frac{1}{z - B_a(z)}\,dz = \frac{1}{1 - B_a'(z_0)}\,.$$

Furthermore, changing variables and using part (3) of Proposition 3.2 we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial_+ D_R^\infty} \frac{1}{z - B_a(z)}\,dz &= -\frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \frac{1}{z^{-1} - B_a(z^{-1})}\frac{1}{z^2}\,dz = \\ = -\frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \frac{1}{z^{-1} - B_{\overline{a}}(z)^{-1}}\frac{1}{z^2}\,dz &= \frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \left(\frac{1}{z - B_{\overline{a}}(z)} - \frac{1}{z}\right)dz\,.\end{aligned} \tag{11}$$

It is not difficult to see that the unique fixed point of $B_{\overline{a}}$ in the unit disk is $\overline{z_0}$ and that $\overline{z_0} \in D_{R^{-1}}$. Moreover, it follows that $B_{\overline{a}}'(\overline{z_0}) = \overline{B_a'(z_0)}$. Thus

$$\frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \left(\frac{1}{z - B_{\overline{a}}(z)} - \frac{1}{z}\right)dz = \frac{1}{1 - \overline{B_a'(z_0)}} - 1\,,$$

and the desired formula follows from Proposition 2.16. □

Since the trace on a Hilbert space is spectral, that is, it coincides with the sum of eigenvalues (see, for example, [12, 4.7.15]), the eigenvalues of a trace class operator L are given by the reciprocals of the zeros of the corresponding spectral determinant $z \mapsto \det(I - zL)$, an entire function given by

$$\det(I - zL) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n}\mathrm{Tr}(L^n)\right)$$

for z in a small neighbourhood of 0 (see, for example, [12, 4.6.2]). We are now able to calculate the eigenvalue sequence of the transfer operator associated with a holomorphically expansive Blaschke product.

Proposition 3.4. *Let B_a be a finite Blaschke product which is holomorphically expansive on $A_{r,R}$. Then*

$$\det(1 - z\mathcal{L}_{B_a}^{\dagger}) = (1 - z) \prod_{k=1}^{\infty} (1 - \mu^k z)(1 - \overline{\mu}^k z)\,,$$

where, as before, μ is the multiplier of the fixed point of B_a in the unit disk. In particular, the eigenvalue sequence of $\mathcal{L}_{B_a}^{\dagger}$ is given by

$$\lambda_n(\mathcal{L}_{B_a}^{\dagger}) = \begin{cases} \mu^{n/2} & \text{for } n \text{ even;} \\ \overline{\mu}^{(n-1)/2} & \text{for } n \text{ odd.} \end{cases}$$

Proof. First we observe that the multiplier of the fixed point in the unit disk of B_a^n, the n-th iterate of B_a, is μ^n. Lemma 3.3 now implies

$$\operatorname{Tr}((\mathcal{L}_{B_a}^{\dagger})^n) = \operatorname{Tr}(\mathcal{L}_{B_a^n}^{\dagger}) = 1 + \frac{\mu^n}{1 - \mu^n} + \frac{\overline{\mu}^n}{1 - \overline{\mu}^n}\,.$$

Thus, for $z \in \mathbb{D}$ we have

$$\begin{aligned}
\log\det(1 - z\mathcal{L}_{B_a}^{\dagger}) &= -\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr}((\mathcal{L}_{B_a}^{\dagger})^n) \\
&= -\sum_{n=1}^{\infty} \frac{z^n}{n} \left(1 + \frac{\mu^n}{1 - \mu^n} + \frac{\overline{\mu}^n}{1 - \overline{\mu}^n}\right) \\
&= -\sum_{n=1}^{\infty} \frac{z^n}{n} - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n}\mu^{kn} z^n - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n}\overline{\mu}^{kn} z^n \\
&= \log(1 - z) + \sum_{k=1}^{\infty} \log(1 - \mu^k z) + \sum_{k=1}^{\infty} \log(1 - \overline{\mu}^k z)\,,
\end{aligned}$$

and the assertions follow. $\square$

Remark 3.5. The proposition above makes it possible to manufacture analytic expanding circle maps of a given degree $d \geq 2$ so that the decay of the eigenvalue sequence of the corresponding transfer operator is exactly exponential. To be precise, let $d \geq 2$ and let $a = (1, 0, a_2, \ldots, a_d)$ with $a_2, \ldots, a_d \neq 0$. Using (5) of Proposition 3.2 it follows that B_a yields an analytic expanding circle map, which is easily seen to be of degree d. Moreover, the unique fixed point of B_a in the unit disk is 0 and the corresponding multiplier $\mu = \prod_{j=2}^{d}(-a_j)$ is non-zero. Thus the above proposition implies that

$$\lim_{n\to\infty} |\lambda_n(\mathcal{L}_{B_a})|^{1/n} = \sqrt{|\mu|}\,.$$

We now turn our attention to anti-Blaschke products, which are defined as follows.

Definition 3.6. If B_a is Blaschke product of degree d then

$$\hat{B}_a(z) = \frac{1}{B_a(z)}\,,$$

will be called an *anti-Blaschke product of degree d* or a *finite anti-Blaschke product.*

Remark 3.7. Note that Blaschke products yield orientation preserving circle maps, while anti-Blaschke products provide examples of orientation reversing circle maps.

For later use, we note the following properties of the second iterate of a finite anti-Blaschke product.

Lemma 3.8. *Let B_a be a Blaschke product and let $\hat{B}_a$ denote the corresponding anti-Blaschke product. If $\hat{B}_a$ is holomorphically expansive, then $\hat{B}_a \circ \hat{B}_a$ is a holomorphically expansive (ordinary) Blaschke product with a unique fixed point $z_0 \in \mathbb{D}$, the multiplier of which satisfies*

$$(\hat{B}_a \circ \hat{B}_a)'(z_0) = |B_a'(z_0)|^2 .$$

Proof. Let B_a be a holomorphically expansive Blaschke product and let $\hat{B}_a$ denote the corresponding holomorphically expansive anti-Blaschke product. We start by observing that by (3) of Proposition 3.2 we have

$$\hat{B}_a(\hat{B}_a(z)) = B_a(B_a(z)^{-1})^{-1} = B_{\overline{a}}(B_a(z)) . \tag{12}$$

Thus, by (4) of Proposition 3.2, the second iterate $\hat{B}_a \circ \hat{B}_a$ is a holomorphically expansive Blaschke product, which, by (6) of Proposition 3.2, has a unique fixed point $z_0 \in \mathbb{D}$.

We shall now show that z_0 is the unique point in $\mathbb{D}$ satisfying

$$B_a(z_0) = \overline{z_0} . \tag{13}$$

In order to see this, note that

$$\begin{aligned} (B_{\overline{a}} \circ B_a)(z_0) &= z_0 \\ (B_a \circ B_{\overline{a}})(B_a(z_0)) &= B_a(z_0) , \end{aligned}$$

which implies that $B_a(z_0) \in \mathbb{D}$ is the unique fixed point in $\mathbb{D}$ of the holomorphically expansive Blaschke product $B_a \circ B_{\overline{a}}$. At the same time we have

$$(B_a \circ B_{\overline{a}})(\overline{z_0}) = B_a(\overline{B_a(z_0)}) = \overline{B_{\overline{a}}(B_a(z_0))} = \overline{z_0} ,$$

so $B_a(z_0) = \overline{z_0}$, as claimed.

Now, using (12) and (13) we see that

$$\begin{aligned} (\hat{B}_a \circ \hat{B}_a)'(z_0) = (B_{\overline{a}} \circ B_a)'(z_0)) &= B_{\overline{a}}'(B_a(z_0))B_a'(z_0) \\ &= B_{\overline{a}}'(\overline{z_0})B_a'(z_0) = \overline{B_a'(z_0)}B_a'(z_0) = |B_a'(z_0)|^2 , \end{aligned}$$

and the remaining claim of the lemma follows. ☐

As for Blaschke products, the analytic structure of anti-Blaschke products makes it possible to calculate the traces of the corresponding transfer operators.

Lemma 3.9. *Let $\hat{B}_a$ be a finite anti-Blaschke product which is a holomorphically expansive on $A_{r,R}$. Then*

$$\mathrm{Tr}(\mathcal{L}^{\dagger}_{\hat{B}_a}) = 1 .$$

Proof. Let $B_a = \hat{B}_a^{-1}$ denote the corresponding Blaschke product. Since $zB_a(z) \in \mathbb{D}$ whenever $z \in \mathbb{D}$, we have

$$\frac{1}{2\pi i}\int_{\partial_+ D_r} \frac{1}{z - B_a(z)^{-1}}\,dz = \int_{\partial_+ D_r} \frac{B_a(z)}{zB_a(z) - 1}\,dz = 0 .$$

Furthermore, changing variables and using (3) of Proposition 3.2 we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial_+ D_R^\infty} \frac{1}{z - B_a(z)^{-1}}\,dz = -\frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \frac{1}{z^{-1} - B_a(z^{-1})^{-1}}\frac{1}{z^2}\,dz = \\ &= -\frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \frac{1}{z^{-1} - B_{\overline{a}}(z)}\frac{1}{z^2}\,dz = \frac{1}{2\pi i}\int_{\partial_+ D_{R^{-1}}} \left(\frac{B_{\overline{a}}(z)}{zB_{\overline{a}}(z) - 1} - \frac{1}{z}\right)dz = -1 . \end{aligned}$$

Since τ is orientation reversing, the assertion follows from Proposition 2.16. $\square$

As before, we are now able to determine the eigenvalue sequence of transfer operators corresponding to anti-Blaschke products.

Proposition 3.10. *Let $\hat{B}_a$ be a finite anti-Blaschke product which is holomorphically expansive on $A_{r,R}$. Then*

$$\det(1 - z\mathcal{L}^{\dagger}_{\hat{B}_a}) = (1-z)\prod_{k=1}^{\infty}(1-\mu^k z)(1+\mu^k z)\,,$$

where $\mu \in [0,1)$ is the square root of the multiplier of the fixed point of $\hat{B}_a \circ \hat{B}_a$ in the unit disk (guaranteed by Lemma 3.8).

In particular, the eigenvalue sequence of $\mathcal{L}^{\dagger}_{\hat{B}_a}$ is given by

$$\lambda_n(\mathcal{L}^{\dagger}_{\hat{B}_a}) = \begin{cases} -\mu^{n/2} & \textit{for } n \textit{ even;} \\ \ \ \mu^{(n-1)/2} & \textit{for } n \textit{ odd.} \end{cases}$$

Proof. We start by observing that by Lemma 3.8 the even iterates of $\hat{B}_a$ are iterates of a finite Blaschke product, while odd iterates are anti-Blaschke products. Moreover, the multiplier of the fixed point in the unit disk of the $2n$-th iterate of $\hat{B}_a$ is μ^{2n}.

Thus, using Lemmas 3.3 and 3.9 it follows that

$$\operatorname{Tr}((\mathcal{L}^{\dagger}_{\hat{B}_a})^n) = \begin{cases} 1 & \text{for } n \text{ odd;} \\ 1 + \frac{2\mu^n}{1-\mu^n} & \text{for } n \text{ even.} \end{cases}$$

Thus, for $z \in \mathbb{D}$ we have

$$\begin{aligned}
\log\det(1 - z\mathcal{L}^{\dagger}_{\hat{B}_a}) &= -\sum_{n=1}^{\infty}\frac{z^n}{n}\operatorname{Tr}((\mathcal{L}^{\dagger}_{\hat{B}_a})^n) \\
&= -\sum_{n=1}^{\infty}\frac{z^n}{n} - \sum_{n=1}^{\infty}\frac{z^{2n}}{2n}\frac{2\mu^{2n}}{1-\mu^{2n}} \\
&= -\sum_{n=1}^{\infty}\frac{z^n}{n} - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\mu^{2nk}z^{2n} \\
&= \log(1-z) + \sum_{k=1}^{\infty}\log(1-\mu^{2k}z^2)\,,
\end{aligned}$$

and the assertions follow. $\square$

Remark 3.11. Arguing as in Remark 3.5, the proposition above allows us to construct orientation reversing analytic expanding circle maps of a given degree $d \leq -2$ so that the decay of the eigenvalue sequence of the corresponding transfer operator is exactly exponential. Let $d \geq 2$ and let $a = (1, 0, a_2, \ldots, a_d)$ with $a_2, \ldots, a_d \neq 0$. As in Remark 3.5, the corresponding Blaschke product B_a is holomorphically expansive, and so is the associated anti-Blaschke product $\hat{B}_a$. Moreover, the unique fixed point of B_a in $\mathbb{D}$ is 0 and the corresponding multiplier is $\prod_{j=2}^{d}(-a_j)$. Since $B_a(0) = B_{\overline{a}}(0) = 0$, equation (12) implies that 0 is the unique fixed point of $\hat{B}_a \circ \hat{B}_a$ in $\mathbb{D}$ and that the corresponding multiplier is given by $\prod_{j=2}^{d}|a_j|^2$. It now follows that the anti-Blaschke product $\hat{B}_a$ is an orientation reversing analytic expanding circle map of degree $-d$ such that the eigenvalues of the corresponding transfer operator satisfy

$$\lim_{n\to\infty} \left|\lambda_n(\mathcal{L}_{\hat{B}_a})\right|^{1/n} = \sqrt{\mu}\,,$$

where $\mu = \prod_{j=2}^{d} |a_j|$.

Remark 3.12. It is rather curious that while the eigenvalues of the transfer operators associated with Blaschke products can have non-vanishing imaginary parts, this is not the case for the eigenvalues of the transfer operators associated with anti-Blaschke products, which are, as the above proposition shows, always real.

4. Potential theoretic tools

In this section we collect some basic definitions and recall, mostly without proofs, the material necessary to prove the main result. Our references are [8] for the theory of several complex variables and [14] for potential theory in the one dimensional case. In our applications, we mainly need to look at the case of one and two complex variables, but we state the results in the n-dimensional case. Let $\mathcal{O} \subset \mathbb{C}^n$ be an open connected non-empty set. We denote by $\Delta(a,r) \subset \mathbb{C}$ the closed Euclidean disc centred at a and of radius r.

Definition 4.1. A real valued function $\varphi : \mathcal{O} \to [-\infty, +\infty)$ is said to be plurisubharmonic on $\mathcal{O}$ if

(1) φ is upper semi-continuous and $\varphi \not\equiv -\infty$ on $\mathcal{O}$.

(2) For all $w \in \mathcal{O}$, for all $r > 0$ and $\zeta \in \mathbb{C}^n$ such that $w + \zeta\Delta(0,r) \subset \mathcal{O}$,

$$\varphi(w) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(w + \zeta r e^{i\theta})\, d\theta.$$

We denote by $\mathrm{PSH}(\mathcal{O})$ the set of plurisubharmonic functions on the domain $\mathcal{O}$. From the above definition one derives the following basic properties (see [8, Appendix 1] for more details).

Proposition 4.2. *Using the above notations, we have the following.*

(1) $\mathrm{PSH}(\mathcal{O})$ *is stable under positive linear combinations.*

(2) *If,* $\varphi_1, \varphi_2 \in \mathrm{PSH}(\mathcal{O})$*, then* $\max\{\varphi_1, \varphi_2\} \in \mathrm{PSH}(\mathcal{O})$*.*

(3) $PSH(\mathcal{O}) \subset L^1_{loc}(\mathcal{O})$*.*

(4) *If* $f : \mathcal{O} \to \mathbb{C}$ *is a non identically zero holomorphic function, then* $\varphi(w) = \log|f(w)|$ *is plurisubharmonic.*

A subset $E \subset \mathcal{O}$ is said to be *pluripolar* if there exists a subharmonic function $\varphi : \mathcal{O} \to [-\infty, +\infty)$ such that $E \subset \{\, w \in \mathcal{O} \,:\, \varphi(w) = -\infty \,\}$. From property (3) of the above Proposition it follows that every pluripolar set is measurable with zero $2n$-dimensional Lebesgue measure. In the one dimensional case one can show (see [14, p. 57]) that every Borel[2] polar set has zero Hausdorff dimension. However, polar sets can be uncountable (see [14, p. 143] for examples of Cantor-like polar sets). One of the key features of plurisubharmonic functions is the following.

[2]Non-Borel pluripolar sets do exist, though they are still Lebesgue measurable. However in our applications we will always encounter Borel sets.

Proposition 4.3. *(Maximum principle) Given $\varphi \in \mathrm{PSH}(\mathcal{O})$, either we have for all $w \in \mathcal{O}$,*

$$\varphi(w) < \sup_{\zeta \in \mathcal{O}} \varphi(\zeta)\,,$$

or $\varphi = \sup_{\mathcal{O}} \varphi$ is a constant.

Let $\mathcal{U} \subset \mathbb{C}$ be a domain and let $\varphi = \varphi(w, \zeta) : \mathcal{U} \times \mathbb{C} \to [-\infty, +\infty)$ be a plurisubharmonic function. For all $w \in \mathcal{U}$ we define the order of growth $\rho_\varphi(w)$ of φ (with respect to ζ) by

$$\rho_\varphi(w) := \limsup_{r \to +\infty} \frac{\log(\sup_{|\zeta| \le r} \max\{\varphi(w, \zeta), 0\})}{\log r}\,.$$

In general, $w \mapsto \rho_\varphi(w)$ is not a subharmonic function so the above maximum principle cannot be applied. However we have the following key result (see [8, p. 25]).

Proposition 4.4. *Assume that $\varphi \in \mathrm{PSH}(\mathcal{U} \times \mathbb{C})$ and that $\varphi \ge 1$. Then for all relatively compact domains $\mathcal{U}' \subset \mathcal{U}$, there exists a sequence of negative functions $\psi_k \in \mathrm{PSH}(\mathcal{U}')$ such that for all $w \in \mathcal{U}'$,*

$$\frac{-1}{\rho_\varphi(w)} = \limsup_{k \to +\infty} \psi_k(w)\,.$$

To prove our main result we also require the following fact (see [8, p. 25]) which serves as a substitute for the maximum principle.

Proposition 4.5. *Let $(\varphi_k)_{k \in \mathbb{N}}$ be a sequence in $\mathrm{PSH}(\mathcal{O})$, uniformly bounded from above on compact subsets. Assume that $\limsup_{k \to +\infty} \varphi_k \le 0$ and that there exists $\xi \in \mathcal{O}$ such that $\limsup_{k \to +\infty} \varphi_k(\xi) = 0$. Then*

$$\limsup_{k \to +\infty} \varphi_k = 0\,,$$

except on a Borel pluripolar subset of $\mathcal{O}$.

5. Complexified homotopies

In this short section we prove a key lemma which will allow us to holomorphically deform an arbitrary analytic expanding map into a suitable finite Blaschke (or anti-Blaschke) product. We start by the so-called lifting lemma in the analytic category, which is a classical result of algebraic topology. However since we will need to specify the domains of holomorphy, we include a proof for completeness.

Lemma 5.1. *Let $f : \mathbb{R} \to \mathbb{T}$ be a real-analytic map. Then there exists $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ real analytic such that for all $x \in \mathbb{R}$*

$$f(x) = e^{i\widetilde{f}(x)}.$$

Proof. Let $\alpha \in \mathbb{R}$ be such that $e^{i\alpha} = f(0)$. For all $x \in \mathbb{R}$, set

$$\widetilde{f}(x) := \frac{1}{i} \int_0^x \frac{f'(t)}{f(t)}\, dt + \alpha.$$

Clearly $\widetilde{f}$ is real analytic and real valued since we have

$$\mathrm{Re}\left(\frac{f'(x)}{f(x)}\right) = \frac{d}{dx}\left(\log |f(x)|\right) = 0\,.$$

Set $g(x) = e^{i\widetilde{f}(x)}$, then g is a solution of the first order linear ODE

$$g' = \frac{f'}{f} g,$$

and thus is proportional to f. Since $g(0) = e^{i\alpha} = f(0)$ we are done. $\square$

Let $P : \mathbb{R} \to \mathbb{T}$ denote the universal covering map given by $P(\theta) = e^{i\theta}$ and let $\tau : \mathbb{T} \to \mathbb{T}$ be an analytic expanding map. By the above lemma, we can always write

$$\tau \circ P(\theta) = e^{i\widetilde{\tau}(\theta)}$$

for some real analytic map $\widetilde{\tau} : \mathbb{R} \to \mathbb{R}$. Then the quantity (independent of θ)

$$\deg(\tau) := \frac{\widetilde{\tau}(\theta + 2\pi) - \widetilde{\tau}(\theta)}{2\pi} \in \mathbb{Z},$$

is called the degree (or winding number) of the map and does not depend on the choice of $\widetilde{\tau}$. The goal of this section is to prove the following.

Lemma 5.2. *Let τ_0, τ_1 be two analytic expanding maps of the circle such that $\deg(\tau_0) = \deg(\tau_1)$. Then there exist two annuli A_{r_0,R_0} and A_{r_1,R_1} with $\mathrm{cl}(A_{r_0,R_0}) \subset A_{r_1,R_1}$, a complex simply connected neighbourhood $\mathcal{U} \supset [0,1]$ and a holomorphic map $T : \mathcal{U} \times \mathrm{cl}(A_{r_0,R_0}) \to \mathbb{C}$ such that:*

1. *for all $w \in \mathcal{U}$, $T(w,.)$ satisfies $T(w, A_{r_0,R_0}) \supset \mathrm{cl}(A_{r_1,R_1})$;*
2. *for all $w \in [0,1]$, $T(w,.)$ is an analytic expanding map of $\mathbb{T}$;*
3. *$T(0,.) = \tau_0$ and $T(1,.) = \tau_1$.*

Proof. Consider

$$\widetilde{T}(w,\theta) := e^{i\{(1-w)\widetilde{\tau}_0(\theta) + w\widetilde{\tau}_1(\theta)\}},$$

and choose $\epsilon > 0$ so small that $\widetilde{T}$ is holomorphic on $\mathbb{C} \times (\mathbb{R} + i[-\epsilon, +\epsilon])$. Let $P(\theta) = e^{i\theta}$ and let A_{r_0,R_0} be given by

$$A_{r_0,R_0} := P(\mathbb{R} + i[-\epsilon, +\epsilon]).$$

We claim, that there exists a unique map $T : \mathbb{C} \times A_{r_0,R_0} \to \mathbb{C}$ such that for all $(w,\theta) \in \mathbb{C} \times (\mathbb{R} + i[-\epsilon, +\epsilon])$, we have

$$T(w, P(\theta)) = \widetilde{T}(w,\theta).$$

Indeed, since $\deg(\tau_0) = \deg(\tau_1)$, we have (by unique continuation) for all $\theta \in \mathbb{R} + i[-\epsilon, +\epsilon]$,

$$\widetilde{\tau}_0(\theta + 2\pi) - \widetilde{\tau}_0(\theta) = \widetilde{\tau}_1(\theta + 2\pi) - \widetilde{\tau}_1(\theta).$$

Therefore for all $(w,\theta) \in \mathbb{C} \times (\mathbb{R} + i[-\epsilon, +\epsilon])$,

$$\widetilde{T}(w, \theta + 2\pi) = \widetilde{T}(w,\theta),$$

which guarantees that T is well defined. Since

$$P : \mathbb{R} + i[-\epsilon, +\epsilon] \to A_{r_0,R_0}$$

is a locally biholomorphic map (a holomorphic covering) it follows that T is a holomorphic map. Remark that by taking ϵ small enough, we can assume that $T(w,.)$ is holomorphic on a neighbourhood of $\mathrm{cl}(A_{r_0,R_0})$. We will now restrict the first variable w to a domain of the type

$$\mathcal{U} := [0,1] + \Delta(0,\eta),$$

where η will be taken small, and $\Delta(0,\eta)$ denotes the complex disc centred at 0 of radius η. To simplify notation further, we set

$$\widetilde{\tau_w}(\theta) := (1-w)\widetilde{\tau_0}(\theta) + w\widetilde{\tau_1}(\theta).$$

Note that since τ_0 and τ_1 are both expanding and have same degree, we have

$$\inf_{(w,\theta)\in[0,1]\times\mathbb{R}} |\partial_\theta \widetilde{\tau_w}(\theta)| > 1.$$

Because $\partial_\theta\widetilde{\tau_w}(\theta)$ is real for $w\in[0,1]$ and θ real, a simple compactness argument shows that it is possible to choose $\eta>0$ and $\epsilon>0$ so small that

$$\rho := \inf_{(w,\theta)\in\mathcal{U}\times\mathbb{R}+i[-\epsilon,+\epsilon]} |\mathrm{Re}\,(\partial_\theta\widetilde{\tau_w}(\theta))\,| > 1.$$

Assuming that $\mathrm{Re}(\partial_\theta\widetilde{\tau_w})>0$, we now observe that

$$\log|T(w,e^{\epsilon+ib})| - \log|T(w,e^{ib})| = \int_0^\epsilon \partial_a \left\{\log|T(w,e^{a+ib})|\right\}\, da$$

$$= \int_0^\epsilon \mathrm{Re}(\partial_\theta\widetilde{\tau_w}(b-ia))\, da \geq \rho\epsilon,$$

while

$$\log|T(w,e^{ib})| - \log|T(w,e^{-\epsilon+ib})| = \int_{-\epsilon}^0 \partial_a \left\{\log|T(w,e^{a+ib})|\right\}\, da \geq \rho\epsilon.$$

Choose $\eta>0$ so small that for all $w\in\mathcal{U}$ and $b\in\mathbb{R}$ we have

$$\log|T(w,e^{ib})| \in [-\overline{\epsilon},+\overline{\epsilon}],$$

where $\overline{\epsilon} = \epsilon\frac{\rho-1}{2}$. We end up with

$$|T(w,e^{\epsilon+ib})| \geq e^{\rho\epsilon-\overline{\epsilon}} = e^{\epsilon(\rho+1)/2} := R > R_0 = e^\epsilon,$$

$$|T(w,e^{-\epsilon+ib})| \leq e^{-\rho\epsilon+\overline{\epsilon}} = e^{-\epsilon(\rho+1)/2} := r < r_0 = e^{-\epsilon}.$$

Therefore, uniformly in $w\in\mathcal{U}$, we have

$$T(w,\mathbb{T}_{r_0}) \subset D_{r_1} \text{ and } T(w,\mathbb{T}_{R_0}) \subset D^\infty_{R_1}$$

for all

$$R_0 < R_1 < R \text{ and all } r < r_1 < r_0.$$

The proof is similar if $\mathrm{Re}(\partial_\theta\widetilde{\tau_w})<0$. □

Note that when $w\notin[0,1]$, the map $z\mapsto T(w,z)$ is *a priori* no longer preserving the unit circle. However, it is still a holomorphically expansive map of some annulus A_{r_0,R_0} in the sense defined previously. An explicit homotopy between $z\mapsto z^2$ and a Blaschke product with non trivial spectrum is provided by

$$T(w,z) = z\left(\frac{2z-w}{2-wz}\right). \tag{14}$$

For a plot of the filled Julia set with $w=0.5+0.26i$, for which the invariant set is a quasi-circle, see Figure 1.

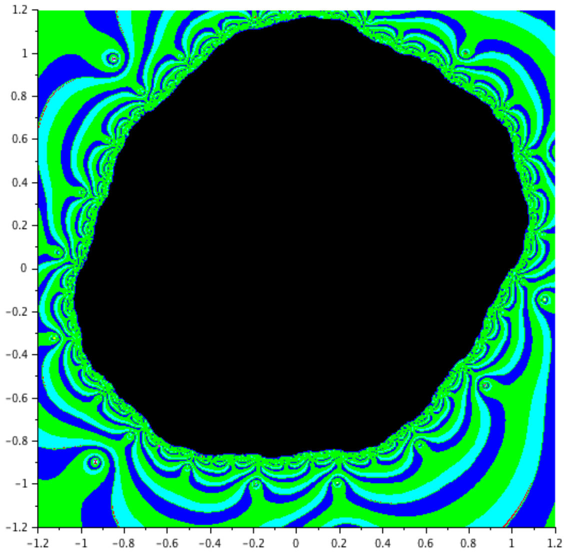

FIGURE 1. Filled Julia set of the map (14) with $w = 0.5 + 0.26i$.

6. Proof of the main result

We start with $\tau : \mathbb{T} \to \mathbb{T}$, analytic expanding, with degree $d \in \mathbb{Z}$, $|d| \geq 2$. We choose a Blaschke product $B : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the same degree d and a non-trivial Ruelle eigenvalue sequence with exponential decay, as in Remark 3.5 and Remark 3.11. Using Lemma 5.2, we have a holomorphic map

$$T : \mathcal{U} \times A_{r,R} \to \mathbb{C}$$

such that $T(0, z) = \tau(z)$ and $T(1, z) = B(z)$ with the property that for each $w \in \mathcal{U}$, the map $z \mapsto T(w, z)$ is holomorphically expansive on the annulus $A_{r,R}$. By Corollary 2.14, we know that the dual operator

$$\mathcal{L}^{\dagger}_{T(w,.)} : H^2(D_r) \oplus H^2_0(D^{\infty}_R) \to H^2(D_r) \oplus H^2_0(D^{\infty}_R)\,,$$

is a compact trace class operator and we consider the determinant

$$\mathcal{Z}(w, \zeta) := \det(I - e^{\zeta}\mathcal{L}^{\dagger}_{T(w,.)})\,,$$

which defines a holomorphic function on $\mathcal{U} \times \mathbb{C}$. Our goal is to investigate the corresponding order function (defined for $w \in \mathcal{U}$)

$$\rho(w) := \limsup_{r\to+\infty} \frac{\log(\sup_{|\zeta|\leq r} \max\{\log |\mathcal{Z}(w, \zeta)|, 0\})}{\log r}\,.$$

We start with the following simple and useful lemma.

Lemma 6.1. *Let $\mathcal{H}$ be a separable complex Hilbert space and let $L : \mathcal{H} \to \mathcal{H}$ be a trace class operator such that the eigenvalue sequence $(\lambda_n(L))_{n\in\mathbb{N}}$ satisfies*

$$|\lambda_n(L)| \leq Ce^{-\alpha n^{\beta}}\,,$$

for some $C, \alpha > 0$ and $\beta \geq 1$. Then, as $|\zeta| \to +\infty$, we have

$$\log|\det(I - e^{\zeta} L)| = O\left(|\zeta|^{\frac{1}{\beta}+1}\right).$$

Proof. Write

$$\begin{aligned}\log|\det(I - e^{\zeta} L)| &\leq \sum_{n\geq 1} \log(1 + Ce^{|\zeta|}e^{-\alpha n^{\beta}}) \\ &\leq N\log(1 + Ce^{|\zeta|}) + Ce^{|\zeta|}\sum_{n\geq N+1} e^{-\alpha n^{\beta}}.\end{aligned}$$

Since

$$\sum_{n\geq N+1} e^{-\alpha n^{\beta}} \leq \int_N^{+\infty} e^{-\alpha t^{\beta}}dt = O(e^{-\alpha N^{\beta}}),$$

setting $N = 2\lfloor(|\zeta|/\alpha)^{1/\beta}\rfloor$ now finishes the proof. □

The first key observation is the following.

Proposition 6.2. *Using the above notations, for all $w \in \mathcal{U}$ we have $\rho(w) \leq 2$ and $\rho(1) = 2$.*

Proof. By Corollary 2.14, we know that $\mathcal{L}^{\dagger}_{T(w,.)}$ is in the exponential class, so it definitely follows from Lemma 6.1 that $\rho(w) \leq 2$. On the other hand, for $w = 1$, we have by Proposition 3.4 and Proposition 3.10 the explicit formula (we state it for the orientation preserving case)

$$\mathcal{Z}(1,\zeta) = (1 - e^{\zeta})\prod_{k=1}^{\infty}(1 - e^{\zeta}\mu^k)(1 - e^{\zeta}\overline{\mu}^k),$$

where $\mu \in \mathbb{D}\setminus\{0\}$. We shall now show that $\rho(1) = 2$. Assume to the contrary that $\rho(1) < 2$ and fix ρ with $\rho(1) < \rho < 2$. Now consider the counting function

$$N(r) := \#\{|\zeta + 1| \leq r \;:\; \mathcal{Z}(1,\zeta) = 0\}.$$

Applying Jensen's formula (see, for example, [14, Chapter 4]), we have (note that $\mathcal{Z}(1,-1) \neq 0$)

$$\int_0^{2R}\frac{N(r)}{r}dr = \frac{1}{2\pi}\int_0^{2\pi}\log|\mathcal{Z}(1,-1+2Re^{i\theta})|\,d\theta - \log|\mathcal{Z}(1,-1)| = O(R^{\rho})$$

as $R \to \infty$. Observing that

$$\int_0^{2R}\frac{N(r)}{r}\,dr \geq \int_R^{2R}\frac{N(r)}{r}\,dr \geq N(R)\int_R^{2R}\frac{1}{r}\,dr = \log(2)N(R),$$

we obtain as $R \to +\infty$,

$$N(R) = O(R^{\rho}).$$

On the other hand, it can be seen from the above explicit product formula that zeros of $\mathcal{Z}(1,\zeta)$ contain a rank 2 lattice, which contradicts the above growth estimate because this implies that $N(R) \geq CR^2$ for all R large by a simple lattice counting argument. □

We are now ready to use Proposition 4.4 and 4.5 to complete the proof. Fix $(\mathcal{U}_n)$ a compact exhaustion of $\mathcal{U}$. For all n large enough, one can find by Proposition 4.4 a sequence $(\psi_k)_{k\in\mathbb{N}}$ of subharmonic functions on $\mathcal{U}_n$ such that

$$\limsup_{k\to\infty}\psi_k(w) = \frac{1}{2} - \frac{1}{\rho(w)} \leq 0.$$

On the other hand, we know that

$$\limsup_{k\to\infty}\psi_k(w)=\frac{1}{2}-\frac{1}{\rho(1)}=0\,,$$

which implies by Proposition 4.5 that $\rho(w)=2$ for all $w\in\mathcal{U}_n\setminus E_n$, where E_n is a polar set. Since a countable reunion of polar sets is polar, we deduce that finally $\rho(w)=2$ for all $w\in\mathcal{U}\setminus E$ where E is a polar set. We know in addition that polar sets have Hausdorff dimension 0 which is more than enough to conclude that $E\cap[0,1]$ has Lebesgue measure 0. Notice that whenever $\rho(w)=2$, we know by Lemma 6.1 that the eigenvalue sequence of $\mathcal{L}^{\dagger}_{T(w,.)}$ has to satisfy

$$\limsup_{n\to\infty}|\lambda_n(\mathcal{L}^{\dagger}_{T(w,.)})|\exp(n^{1+\epsilon})>0\,,$$

for every $\epsilon>0$, otherwise it would produce a contradiction. The very end of the proof follows from the observation that for all compact subset $K\subset\mathcal{U}$ and all

$$r<r_1<1<R_1<R,$$

we have for all η small,

$$\sup_{z\in A_{r_1,R_1}}|T(0,z)-T(\eta,z)|\leq\eta\sup_{K\times A_{r_1,R_1}}|\partial_1 T|\,.$$

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