Lagrange approximation of transfer operators associated with holomorphic data

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Abstract. We show that spectral data of transfer operators given by holomorphic data can be approximated using an effective numerical scheme based on Lagrange interpolation. In particular, we show that for one-dimensional systems satisfying certain complex contraction properties, spectral data of the approximants converge exponentially to the spectral data of the transfer operator with the exponential rate determined by the respective complex contraction ratios of the underlying systems. We demonstrate the effectiveness of this scheme by numerically computing eigenvalues of transfer operators arising from interval and circle maps, as well as Lyapunov exponents of (positive) random matrix products and iterated function systems, based on examples taken from the literature.

1. Introduction

Transfer operators constitute powerful tools for analysing the behaviour of hyperbolic dynamical systems, as their spectral data encode various dynamic and geometric quantities of interest, such as invariant measures, exponential mixing rates, zeta functions counting periodic orbits, resonances of hyperbolic surfaces, escape rates, stationary probability measures of iterated function systems, or Lyapunov exponents of random matrix products (see [Rue, BoyG, Bal1, BahBF, Bor, Bal2] for some pointers to the vast literature on this subject).

Unfortunately, spectral data of transfer operators, which typically act on infinite-dimensional spaces, is rarely available explicitly (see, however, [BanJS, SBJ2, FGL] for examples with explicit spectra). As a result, one needs to construct a suitable discretisation of the operator in the form of a matrix and numerically compute the corresponding eigenvalues and eigenfunctions. A popular approach is to use a projection-based method, also known as 'finite section method' or 'Galerkin method' in various other contexts: given a bounded operator $L\colon B \to B$ on a Banach space B, one considers an approximation scheme determined by a sequence of rank-n projections $(\Pi_n\colon B \to B)_{n\in\mathbb{N}}$ converging to the identity in a suitable sense, and then takes spectral data of $\Pi_n L \Pi_n$, represented by an $n \times n$ matrix, as an approximation to spectral data of L. However, unless extra conditions are imposed on $(\Pi_n)_{n\in\mathbb{N}}$ and L, spectral data of $(\Pi_n L \Pi_n)_{n\in\mathbb{N}}$ need not converge to spectral data of L, see, for example, [Ha].

The most widely studied projection-based method for the spectral approximation of transfer operators is the Ulam method, originally proposed in [U] as a means to compute the fixed point of the Perron-Frobenius operator for expanding interval maps, which yields the density of the unique absolutely continuous invariant measure for the underlying map. The Ulam method is based on partitioning the phase space of the underlying dynamical system into n disjoint sets and defining Π_n to be the conditional expectation with respect to this partition, that is, Π_n is the projection onto the

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subspace spanned by functions which are piecewise constant on this partition. The convergence of the Ulam method for piecewise monotonic and expanding interval maps was established in [Li]. The rate of convergence was proven to be $O(\log(n)/n)$ in [Ke, BlK], which in turn was shown to be optimal in [BosM], even for systems with higher regularity.

There exists by now a considerable body of literature concerned with the application of Ulam-type methods to approximate the leading eigenvalue and eigenfunction as well as subleading eigenvalues of transfer operators, including those arising from higher-dimensional expanding or hyperbolic maps, see [F1, BlK, DJ, BlKL, F2, BahB, FG-TQ, GN, CrF1] to name but a few. For dynamical system with higher regularity, the speed of convergence of projection-based methods can be improved by choosing projections onto subspaces spanned by functions of higher smoothness, see [Liv] for a discussion of a general strategy or [BalH] for an approach using wavelets. For practical applications, however, these methods are often less suitable, as they typically involve numerical evaluation of integrals as well as higher derivatives.

In this article, we shall study a spectral approximation scheme based on interpolating projections, which, for transfer operators associated with holomorphic data, yields exponential convergence of spectral data, while remaining practically efficient. In this scheme, the projection operator Π_n maps a function f to the unique (Laurent) polynomial of degree n that coincides with f on a set of n predefined collocation points. The resulting approximation scheme, variously known in other contexts as Lagrange interpolation, spectral Galerkin or collocation method, is easily implementable in practice, as the n-th approximant $\Pi_n\mathcal{L}\Pi_n$ of the transfer operator $\mathcal{L}$ can be obtained from an $n\times n$ matrix of the form $((\mathcal{L}e_j)(x_i))_{ij}$, where $(e_j)_j$ is a suitable collection of (Laurent) polynomials and $(x_i)_i$ a suitable collection of collocation points (both of which may depend on n). This method has recently been applied to transfer operators for expanding circle or interval maps by Wormell [W]; see also [BanPSW], where this method has been used to effectively calculate resonances of Schottky surfaces using transfer operator methods. The analysis in [W] is in fact based on a different projection scheme, in which the rank-n projection is chosen to be the orthogonal projection (with respect to a suitably weighted L^2-scalar product) onto the first n elements of a Fourier basis (for circle maps) or a Chebyshev basis (for interval maps), an idea that is already present in an earlier paper by MacKernan and Basios [MB]. The main result of [W] shows that the leading eigenfunction of the transfer operator can be approximated exponentially fast in bounded variation norm by the leading eigenfunctions of the transfer operators truncated using the above orthogonal projection scheme. The paper also provides an algorithm in which the L^2-inner products arising in the matrices representing the truncated transfer operator are evaluated using Chebyshev–Gauss quadrature, which effectively renders the described algorithm into a Lagrange interpolation method, where the resulting errors are controlled using certain aliasing bounds.

By contrast, we directly investigate the (non-orthogonal) Lagrange interpolation projection, which allows us to obtain uniform convergence of the approximation scheme on suitable Banach spaces of holomorphic functions, which in turn yields strong convergence results for all spectral data, including eigenvalues as well as the corresponding generalised eigenfunctions and eigendistributions. Moreover, we obtain explicit bounds on the exponential convergence rate given by certain complex contraction ratios of the underlying system. Whereas [MB, W] are chiefly concerned with Perron-Frobenius operators for expanding (Markov) maps, our results apply to a more general class of transfer operators associated to so-called holomorphic map-weight systems (see, for example, [BanJ1, BanJ2]).

We should also mention that there is a completely different approach for approximating statistical properties of dynamical systems based on an intimate relation between certain Fredholm determinants encoding eigenvalues of transfer operators and periodic orbits of the underlying systems. This method became popular with the papers [ArAC1, ArAC2] and was investigated rigorously for approximating invariant densities and metric entropy of analytic expanding Markov maps [PoJ], computing Lyapunov exponents of random matrix products of positive matrices [Po, JM], as well as other dynamical quantities and invariants [JP1, JP2, BanJP, JP4, JPV, CJ]. The convergence of these periodic orbit based algorithms is super-exponential in the maximum period of periodic orbits used. However, the number

of periodic orbits grows exponentially with the period, rendering their computation intractable for large periods. Moreover, the expression involving periodic orbits used in these algorithms can become numerically unstable for transfer operators with large weights, see the discussion in [BanPSW].

The structure of the paper is as follows. We begin with a brief introduction to the Lagrange interpolation problem on the unit circle (Section 2.1). We then specialise to the case of equidistant interpolation, and, in Section 2.2, prove a key lemma for the error of approximation of the interpolation operator when considered on Hardy spaces of bounded holomorphic functions on certain annuli containing the unit circle. In Section 2.3, we review the interpolation problem on the interval $[-1, 1]$ with the interpolation points chosen as zeros of Chebyshev polynomials, and show that the corresponding Lagrange–Chebyshev interpolation operator considered on the Hardy space of bounded holomorphic functions on Bernstein ellipses is isometrically isomorphic to the equidistant interpolation operator from Section 2.2, yielding the same bounds for the error of approximation. In Section 2.4 the results from the previous section are extended to Hardy spaces of bounded holomorphic functions on more general (confocal) ellipses. In Section 3.1, we study transfer operators associated to holomorphic map-weight systems, and show (Theorem 3.3) that these can be approximated exponentially fast in operator norm by finite-rank Lagrange–Chebyshev approximants. An analogous result for generalised transfer operators arising from analytic expanding circle maps (Theorem 3.7) is presented in Section 3.2. The resulting exponential convergence of spectral data, obtained as a straightforward consequence of our main results, Theorem 3.3 and Theorem 3.7, are collected as Corollaries 3.9 and 3.11 in Section 3.3. Finally, in Section 4, we demonstrate that a practical algorithm based on Lagrange–Chebyshev approximation can be used to effectively compute spectral data of suitable transfer operators, by applying it to several examples from the literature and comparing it to other approximation methods. These include eigenvalues of transfer operators arising from interval or circle maps, as well as Lyapunov exponents for random matrix products and iterated functions systems.

2. Lagrange interpolation

2.1. Lagrange interpolation on the unit circle. We start by recalling basic facts concerning the Lagrange interpolation problem on the unit circle (see, for example, [DG-V]). For a brief overview over interpolation in general, see, for example, [Riv, Chapter 4].

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, and let $f\colon \mathbb{T} \to \mathbb{C}$ be a continuous function. For $n \in \mathbb{Z}$ let e_n denote the Laurent monomial $e_n(z) = z^n$. We are interested in approximating f by Laurent polynomials, that is, finite linear combinations of Laurent monomials.

For $N \in \mathbb{N}$, let $\Gamma = \{z_k\colon k = 0, \dots, N-1\} \subset \mathbb{T}$ be a set of N distinct complex numbers on the unit circle, and let N_l and N_u be two non-negative integers with $N_l + N_u = N - 1$. We shall refer to the points in Γ as interpolation nodes, or simply nodes. The Lagrange interpolation problem in the space of Laurent polynomials $\Lambda_{-N_l,N_u} = \operatorname{span}\{e_n\colon -N_l \le n \le N_u\}$ amounts to determining the unique Laurent polynomial $q_N \in \Lambda_{-N_l,N_u}$ with

$$q_N(z_k) = f(z_k) \text{ for } k = 0, \dots, N-1.$$

It is not difficult to see that the interpolant q_N can be written as a linear combination of Lagrange Laurent polynomials as follows

$$q_N(z) = \sum_{k=0}^{N-1} f(z_k) l_k(z). \tag{1}$$

Here, l_k is the unique Laurent polynomial in Λ_{-N_l,N_u} satisfying $l_k(z_j) = \delta_{jk}$, which is given by

$$l_k(z) = \frac{l(z) z_k^{N_l}}{l'(z_k)(z - z_k) z^{N_l}},$$

where $l(z) = \prod_{k=0}^{N-1}(z - z_k)$. Note that (1) defines a projection operator on the space of continuous functions, denoted by $Q_N\colon C(\mathbb{T}) \to C(\mathbb{T})$ with $q_N = Q_N f$.

As we are interested in applications to transfer operators arising from analytic maps, we shall study the operator Q_N when acting on functions analytic on $\mathbb{T}$, which have analytic extensions to certain domains containing $\mathbb{T}$. With slight abuse of notation we continue to write f for various extensions as well. An important ingredient for the proofs presented in the following subsections is the observation that the sum in (1) can be rewritten as a contour integral over such domains. More precisely, assuming that f is holomorphic on the closure of an annulus A containing $\mathbb{T}$, the Residue Theorem implies that

$$f(z) - Q_N f(z) = \frac{1}{2\pi i} \int_{\partial A} \frac{f(\zeta) l(z) \zeta^{N_l}}{(\zeta - z) l(\zeta) z^{N_l}} \, d\zeta \qquad (z \in A), \tag{2}$$

$$Q_N f(z) = \frac{1}{2\pi i} \int_{\partial A} \frac{f(\zeta) \left(l(\zeta) z^{N_l} - l(z) \zeta^{N_l} \right)}{(\zeta - z) l(\zeta) z^{N_l}} \, d\zeta \qquad (z \in A), \tag{3}$$

where ∂A denotes the positively oriented boundary of A. In order to see this, first note that $\frac{z_k^{N_l}}{l'(z_k)(z-z_k)}$ is the residue of $\zeta \mapsto \frac{\zeta^{N_l}}{l(\zeta)(z-\zeta)}$ at the simple pole $\zeta = z_k$ for any $z \in A$ with $z \neq z_k$. For any such z, the Residue Theorem allows us to replace the sum in (1) by a contour integral

$$q_N(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta) l(z) \zeta^{N_l}}{(\zeta - z) l(\zeta) z^{N_l}} \, d\zeta,$$

where $\mathcal{C}$ is a simple closed positively oriented contour in A enclosing all z_k, $k = 0, \ldots, N-1$ but not z. Enlarging the contour of integration and accounting for the residue at z, Equation (2) follows, which in turn implies Equation (3). The expression in (3) is also known as the Hermite integral formula (see, for example, [T, Theorem 11.1]).

2.2. Equidistant interpolation on the unit circle. We shall now specialise the Lagrange interpolation problem to nodes equally spaced on $\mathbb{T}$. For this, we set $N = 2n$, fix $N_l = n$, $N_u = n-1$, and choose the $2n$ equidistant nodes as roots of -1, that is,

$$z_k = \exp\left(\frac{2k+1}{2n} i\pi \right) \qquad (k = 0, \ldots, 2n-1), \tag{4}$$

so that $l(z) = z^{2n} + 1$. We may now rewrite the Laurent polynomial l_k in (1) as follows

$$l_k(z) = \frac{z^n + z^{-n}}{2n z_k^{n-1} (z - z_k)} = \frac{1}{2n} \sum_{l=-n}^{n-1} \left(\frac{z}{z_k} \right)^l.$$

Now, using the above expression for l_k and rearranging the sum in (1) we obtain the following representation of the equidistant interpolation projection $Q_N = Q_{2n}$

$$(Q_{2n} f)(z) = \frac{1}{2n} \sum_{l=-n}^{n-1} c_{l,2n}(f) e_l(z) \qquad (z \in \mathbb{T}, f \in C(\mathbb{T})), \tag{5}$$

where the coefficient functionals

$$c_{l,2n}(f) = \sum_{k=0}^{2n-1} f(z_k) z_k^{-l} \qquad (f \in C(\mathbb{T}))$$

turn out to be the discrete Fourier transform of the sequence $f(z_0), f(z_1), \ldots, f(z_{2n-1})$. As a result, the expression (5) is particularly useful for numerical implementation. In the case where f is not only continuous but also holomorphic on the closure of an annulus A containing $\mathbb{T}$, the representations (2) and (3) take the following form

$$f(z) - (Q_{2n} f)(z) = \frac{1}{2\pi i} \int_{\partial A} \frac{f(\zeta) \sigma(z^n)}{(\zeta - z) \sigma(\zeta^n)} \, d\zeta \qquad (z \in A), \tag{6}$$

$$(Q_{2n}f)(z) = \frac{1}{2\pi i}\int_{\partial A} \frac{f(\zeta)\,(\sigma(\zeta^n)-\sigma(z^n))}{(\zeta - z)\sigma(\zeta^n)}\,d\zeta \qquad (z \in A), \tag{7}$$

where we have used $l(z)/z^n = (z^n + z^{-n}) = 2\sigma(z^n)$ with σ denoting the Joukowski transform

$$\sigma(z) = \frac{1}{2}(z + z^{-1}). \tag{8}$$

We shall now prove a simple lemma that will turn out to be the key estimate allowing us to establish convergence of the Lagrange interpolation method for transfer operators arising from expanding analytic circle maps. In order to formulate it we require some more notation.

For $\rho > 1$ let $A_\rho = \{z \in \mathbb{C}\colon \rho^{-1} < |z| < \rho\}$ be an open annulus with radii ρ and ρ^{-1}. We write $H^\infty(A_\rho)$ for the Hardy space of bounded holomorphic functions on A_ρ which, equipped with the norm $\|f\|_{H^\infty(A_\rho)} = \sup\{|f(z)|\colon z \in A_\rho\}$, is a Banach space.

Observe now that for $1 < r < R$, we have $A_r \subset A_R$, so $H^\infty(A_R)$ can be identified with a subspace of $H^\infty(A_r)$ via the canonical embedding $J\colon H^\infty(A_R) \to H^\infty(A_r)$ given by $Jf = f|_{A_r}$. The following lemma shows that J is approximated at exponential speed by the equidistant Lagrange interpolation projections.

2.1. Lemma. *For $1 < r < R$ let $J\colon H^\infty(A_R) \to H^\infty(A_r)$ denote the canonical embedding and consider Q_{2n} as an operator from $H^\infty(A_R)$ to $H^\infty(A_r)$. Then, for any $n \in \mathbb{N}$,*

$$\|J - Q_{2n}\|_{H^\infty(A_R)\to H^\infty(A_r)} \le c_{r,R}\,\frac{\cosh(n\log(r))}{\sinh(n\log(R))}, \tag{9}$$

$$\|Q_{2n}\|_{H^\infty(A_R)\to H^\infty(A_r)} \le c_{r,R}\,\frac{\cosh(n\log(R)) + \cosh(n\log(r))}{\sinh(n\log(R))}, \tag{10}$$

where

$$c_{r,R} = \frac{\sinh(\log R)}{\cosh(\log(R)) - \cosh(\log(r))}.$$

Proof. We start with the following simple inequalities:

(i) $|\sigma(z^n)| \le \frac{1}{2}(r^n + r^{-n})$ for any $z \in A_r$ and any $n \in \mathbb{N}$;

(ii) $|\sigma(\zeta^n)| \ge \frac{1}{2}(R^n - R^{-n})$ for any $\zeta \in \partial A_R$ and any $n \in \mathbb{N}$.

Next, fix $f \in H^\infty(A_R)$ with $\|f\|_{H^\infty(A_R)} \le 1$ and let $z \in A_r$. By [Rud, Theorem 17.11], the (non-tangential) limit $f^*(\zeta)$ for $\zeta \in \partial A_R$ exists a.e. and f^* is integrable on ∂A_R. Moreover, the integral representations (6) and (7) remain valid with ∂A_R in place of ∂A. Using the inequalities (i) and (ii) above, we thus obtain

$$\begin{aligned}|f(z) - Q_{2n}f(z)| &\le \frac{1}{2\pi}\int_{|\zeta|=R} \frac{|f^*(\zeta)||\sigma(z^n)|}{|z-\zeta||\sigma(\zeta^n)|}\,|d\zeta| + \frac{1}{2\pi}\int_{|\zeta|=R^{-1}} \frac{|f^*(\zeta)||\sigma(z^n)|}{|z-\zeta||\sigma(\zeta^n)|}\,|d\zeta| \\ &\le \left(\frac{R}{R-|z|} + \frac{R^{-1}}{|z|-R^{-1}}\right)\left(\frac{r^n + r^{-n}}{R^n - R^{-n}}\right) = \left(\frac{R - R^{-1}}{(R+R^{-1}) - (|z| + |z|^{-1})}\right)\left(\frac{r^n + r^{-n}}{R^n - R^{-n}}\right),\end{aligned}$$

which, after invoking inequality (i) once more, yields

$$|f(z) - Q_{2n}f(z)| \le \frac{\sinh(\log R)}{\cosh(\log(R)) - \cosh(\log(r))}\,\frac{\cosh(n\log(r))}{\sinh(n\log(R))},$$

which in turn furnishes (9).

Similarly, for $f \in H^\infty(A_R)$ with $\|f\|_{H^\infty(A_R)} \le 1$ and $z \in A_r$ we obtain

$$|Q_{2n}f(z)| \le \left(\frac{R - R^{-1}}{(R+R^{-1}) - (r + r^{-1})}\right)\left(\frac{R^n + R^{-n} + r^n + r^{-n}}{R^n - R^{-n}}\right),$$

from which (10) follows. □

2.2. Remark. In particular, Lemma 2.1 implies that

$$\|J - Q_{2n}\|_{H^\infty(A_R)\to H^\infty(A_r)} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n \to \infty,$$
$$\|Q_{2n}\|_{H^\infty(A_R)\to H^\infty(A_r)} = O(1) \text{ as } n \to \infty,$$

that is, equidistant Lagrange interpolation is stable and converges exponentially to the canonical embedding of $H^\infty(A_R)$ in $H^\infty(A_r)$.

2.3. Lagrange–Chebyshev interpolation on the interval $[-1,1]$. Lagrange interpolation on the unit circle at equidistant nodes is closely related to Lagrange–Chebyshev interpolation on the interval $[-1,1]$, as we shall see presently.

We write T_n for the Chebyshev polynomial of the first kind of degree n, which is given by $T_n(\cos(\theta)) = \cos(n\theta)$ for $n \in \mathbb{N}_0$. The zeros of T_n, referred to as Chebyshev nodes (of order n), are the orthogonal projections of the $2n$ equidistant nodes in (4) onto the interval $[-1,1]$, and are given by

$$x_k = \cos\left(\frac{(2k+1)\pi}{2n}\right) \quad (k = 0, \dots, n-1). \tag{11}$$

Suppose now that we are given a continuous function $f\colon [-1,1] \to \mathbb{C}$. The Lagrange–Chebyshev interpolation problem is to find the unique polynomial $P_n f$ of degree $n-1$ that coincides with f at the Chebyshev nodes $x_0, \dots, x_{n-1}$ of order n. This polynomial can be written

$$(P_n f)(x) = \sum_{k=0}^{n-1} f(x_k) l_k(x),$$

where l_k is the Lagrange polynomial associated with x_k, given by

$$l_k(x) = \frac{T_n(x)}{T_n'(x)(x - x_k)}.$$

The resulting projection operator $P_n\colon C([-1,1]) \to C([-1,1])$ will be referred to as Lagrange–Chebyshev projection operator. Following the same arguments as in Section 2.1, for f extending holomorphically to a complex neighborhood $U \supset [-1,1]$, it is easy to see that P_n has the representation

$$(P_n f)(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)(T_n(\zeta) - T_n(z))}{(\zeta - z)T_n(\zeta)}\, d\zeta \qquad (z \in U), \tag{12}$$

where $\mathcal{C}$ is a simple closed positively oriented contour in U containing z and the interval $[-1,1]$ in its interior.

The operators P_n and Q_{2n} are intimately related, and, as we shall see in Lemma 2.6, the convergence properties of the former can be deduced from those of the latter. Before establishing this, we require some more terminology.

2.3. Notation. For $\rho \in \mathbb{R}$ with $\rho > 1$ we write

$$E_\rho := \left\{\frac{1}{2}(w + w^{-1}) : w \in A_\rho\right\}$$

for the domain in $\mathbb{C}$ containing the origin, bounded by the ellipse with lengths of major and minor semi-axis given by $a = \cosh(\log\rho)$ and $b = \sinh(\log\rho)$, respectively, and foci at -1 and 1. We shall refer to E_ρ as a *standard elliptic domain*, or, in slight abuse of terminology, simply as a *standard ellipse*. Note also that we have

$$E_\rho = \sigma(A_\rho).$$

To each E_ρ we associate the Hardy space $H^\infty(E_\rho)$, that is, the Banach space of bounded holomorphic functions on E_ρ equipped with the norm $\|f\|_{H^\infty(E_\rho)} = \sup\{|f(z)|\colon z \in E_\rho\}$. We shall now show that there is an isometric isomorphism between $H^\infty(E_\rho)$ and the Banach space

$$H_e^\infty(A_\rho) = \{f \in H^\infty(A_\rho)\colon f(z) = f(1/z),\ \forall z \in A_\rho\}.$$

2.4. Lemma. *Let D_σ be the composition operator given by*

$$D_\sigma\colon f \mapsto f \circ \sigma.$$

Then, D_σ is an isometric isomorphism between $H^\infty(E_\rho)$ and $H_e^\infty(A_\rho)$.

Proof. As σ is holomorphic on A_ρ with $\sigma(A_\rho) \subseteq E_\rho$, it follows that $D_\sigma f$ is holomorphic on A_ρ for any $f \in H^\infty(E_\rho)$. Since $\sigma(z) = \sigma(1/z)$ it follows that $(D_\sigma f)(z) = (D_\sigma f)(1/z)$. Note that σ is surjective (more precisely it is a two-to-one map from A_ρ to E_ρ), which implies that D_σ is an isometry, since

$$||D_\sigma f||_{H_e^\infty(A_\rho)} = \sup_{z\in A_\rho} |(f\circ\sigma)(z)| = \sup_{w\in\sigma(A_\rho)} |f(w)| = \sup_{w\in E_\rho} |f(w)| = \|f\|_{H^\infty(E_\rho)}.$$

Moreover, the surjectivity of σ implies that D_ρ is injective.

In order to show that D_σ is surjective, we pick $g \in H_e^\infty(A)$ and write it as a Laurent series $g(z) = \sum_{n=-\infty}^{\infty} c_n e_n(z)$. Since $g(z) = g(1/z)$ it follows that $c_n(g) = c_{-n}(g)$ for all $n \in \mathbb{N}$, and so

$$g(z) = c_0(g) + 2\sum_{n=1}^{\infty} c_n(g)(\sigma(e_n(z)) = c_0(g) + 2\sum_{n=1}^{\infty} c_n(g)T_n(\sigma(z)).$$

Now, as this series converges uniformly on compact subsets of A_ρ, the surjectivity of σ implies uniform convergence of $c_0(g) + 2\sum_{n=1}^{\infty} c_n(g)T_n(w)$ on compact subsets of E_ρ, implying that $f(w) = c_0(g) + 2\sum_{n=1}^{\infty} c_n(g)T_n(w)$ is holomorphic on E_ρ. As $\sup_{w\in E_\rho} |f(w)| = \|g\|_{H_e^\infty(A_\rho)}$ we have an $f \in H^\infty(E_\rho)$ satisfying $g = D_\sigma f$. □

Using the fact that $\sigma(z) = \sigma(1/z)$ for any $z \in A_\rho$, we can consider the operator Q_{2n} given in (7) as an operator from $H_e^\infty(A_R)$ to $H_e^\infty(A_r)$ for some $1 < r < R$.

2.5. Lemma. *For $1 < r < R$, consider Q_{2n} in (7) as an operator from $H_e^\infty(A_R)$ to $H_e^\infty(A_r)$, and consider P_n in (12) as an operator from $H^\infty(E_R)$ to $H^\infty(E_r)$. Denote by $D_{\sigma,q}\colon H^\infty(E_q) \to H_e^\infty(A_q)$ with $q = r, R$, the isomorphisms defined in Lemma 2.4. Then,*

$$Q_{2n}D_{\sigma,R} = D_{\sigma,r}P_n.$$

Proof. First observe that we have the following relation

$$\frac{\zeta}{\zeta - z} - \frac{\zeta^{-1}}{\zeta^{-1} - z} = \frac{1}{1 - z\zeta^{-1}} + \frac{z^{-1}}{\zeta - z^{-1}} = \frac{\zeta - \zeta^{-1}}{(1 - z\zeta^{-1})(\zeta - z^{-1})} = \frac{\sigma'(\zeta)\zeta}{\sigma(\zeta) - \sigma(z)}, \tag{13}$$

where ζ and z are any non-zero complex numbers. For any such z we also have the relation $T_n(\sigma(z)) = \frac{1}{2}(z^n + z^{-n}) = \sigma(e_n(z)) = \sigma(e_{-n}(z))$. Using the shorthand

$$K(\zeta, z) = \frac{\sigma(\zeta^n) - \sigma(z^n)}{\sigma(\zeta^n)} = \frac{T_n(\sigma(\zeta)) - T_n(\sigma(z))}{T_n(\sigma(\zeta))},$$

we obtain, for any $f \in H^\infty(E_R)$, any $z \in A_r$ and any ρ with $r < \rho < R$

$$\begin{aligned}
Q_{2n}(D_{\sigma,R}f)(z) &= \frac{1}{2\pi i}\int_{\partial A_\rho} \frac{f(\sigma(\zeta))K(\zeta,z)}{\zeta - z}\,d\zeta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\sigma(\rho e^{it}))K(\rho e^{it},z)\rho e^{it}}{\rho e^{it} - z}\,dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\sigma((\rho e^{it})^{-1}))K((\rho e^{it})^{-1},z)}{((\rho e^{it})^{-1} - z)\rho e^{it}}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\sigma(\rho e^{it}))K(\rho e^{it},z)\sigma'(\rho e^{it})\rho e^{it}}{\sigma(\rho e^{it}) - \sigma(z)}\,dt \\
&= \frac{1}{2\pi i}\int_{\partial E_\rho} \frac{f(\zeta)(T_n(\zeta) - T_n(\sigma(z)))}{(\zeta - \sigma(z))T_n(\zeta)}\,d\zeta \\
&= D_{\sigma,r}(P_n f)(z),
\end{aligned}$$

where the third equality uses relation (13), and the penultimate equality follows from a change of variables with $\zeta = \sigma(\rho e^{it})$. □

The convergence properties of P_n now follow from the convergence properties of Q_{2n}.

2.6. Lemma. *Let $1 < r < R$ and let $\hat{J}\colon H^\infty(E_R) \to H^\infty(E_r)$ denote the canonical embedding. Then, for any $n \in \mathbb{N}$,*

$$\|\hat{J} - P_n\|_{H^\infty(E_R)\to H^\infty(E_r)} \le c_{r,R}\,\frac{\cosh(n\log(r))}{\sinh(n\log(R))}, \tag{14}$$

$$\|P_n\|_{H^\infty(E_R)\to H^\infty(E_r)} \le c_{r,R}\,\frac{\cosh(n\log(R)) + \cosh(n\log(r))}{\sinh(n\log(R))}, \tag{15}$$

where

$$c_{r,R} = \frac{\sinh(\log R)}{\cosh(\log(R)) - \cosh(\log(r))}.$$

Proof. Let $D_{\sigma,r}$ and $D_{\sigma,R}$ denote the isometric isomorphisms given in Lemma 2.4 and observe that $\hat{J} = D_{\sigma,r}^{-1} J D_{\sigma,R}$ and $P_n = D_{\sigma,r}^{-1} Q_{2n} D_{\sigma,R}$. The bounds (14) and (15) now follow from Lemma 2.1. □

2.7. Remark. For practical purposes (see the algorithm in Section 4), a more useful representation of the Lagrange–Chebyshev interpolation operator P_n is given by (see, for example, [E])

$$(P_n f)(x) = \frac{d_{0,n}(f)}{2} + \sum_{l=1}^{n-1} d_{l,n}(f) T_l(x) \qquad (x \in [-1,1], f \in C([-1,1])), \tag{16}$$

where

$$d_{l,n}(f) = \frac{2}{n}\sum_{k=0}^{n-1} f(x_k) T_l(x_k) \qquad (f \in C([-1,1])),$$

which can be derived from expression (5) and the fact that $Q_{2n} D_\sigma = D_\sigma P_n$.

2.4. Lagrange–Chebyshev approximation of holomorphic functions on general elliptic domains. In the following, we generalise the results of the previous subsection to functions holomorphic on domains bounded by ellipses with arbitrary (distinct) foci $\gamma_+, \gamma_- \in \mathbb{C}$.

For $\gamma = (\gamma_+, \gamma_-)$, define a linear map $\alpha_\gamma\colon \mathbb{C} \to \mathbb{C}$ as

$$\alpha_\gamma(z) = \frac{\gamma_+ - \gamma_-}{2} z + \frac{\gamma_+ + \gamma_-}{2}.$$

Then

$$E_{\gamma,\rho} = \alpha_\gamma(E_\rho)$$

is a domain bounded by an ellipse with foci at γ_+ and γ_-. For $\gamma = (1,-1)$ we recover the standard elliptic domain E_ρ.

Let $H^\infty(E_{\gamma,\rho})$ denote the Banach space of bounded holomorphic functions on $E_{\gamma,\rho}$ equipped with the supremum norm. We have the following simple lemma.

2.8. Lemma. *Let C_{α_γ} be the composition operator given by*

$$C_{\alpha_\gamma}\colon f \mapsto f \circ \alpha_\gamma.$$

Then, C_{α_γ} is an isometric isomorphism between $H^\infty(E_{\gamma,\rho})$ and $H^\infty(E_\rho)$.

Proof. As a_γ is invertible, it follows that C_{α_γ} is invertible with inverse $C_{\alpha_\gamma}^{-1} = C_{\alpha_\gamma^{-1}}$. Also, for any $f \in H^\infty(E_{\gamma,\rho})$ we have

$$\left\|C_{\alpha_\gamma} f\right\|_{H^\infty(E_\rho)} = \sup_{z\in E_\rho} |(f\circ\alpha_\gamma)(z)| = \sup_{w\in\alpha_\gamma(E_\rho)} |f(w)| = \|f\|_{H^\infty(E_{\gamma,\rho})},$$

so C_{α_γ} is indeed an isometric isomorphism. □

Using the previous lemma we can now define generalised Lagrange–Chebyshev projections $P_{\gamma,n}$ on $H^\infty(E_{\gamma,R})$ by

$$P_{\gamma,n} = C_{\alpha_\gamma}^{-1} P_n C_{\alpha_\gamma}. \tag{17}$$

For f a continuous function on the line segment $\alpha_\gamma([-1,1])$, $P_{\gamma,n}f$ is the polynomial of degree $n-1$ that coincides with f at the images of the Chebyshev points of order n under α_γ. The following is a straightforward consequence of Lemma 2.6 and the previous lemma.

2.9. Lemma. *Let $\hat{J}_\gamma \colon H^\infty(E_{\gamma,R}) \to H^\infty(E_{\gamma,r})$ denote the canonical embedding with $1 < r < R$. Then, for any $n \in \mathbb{N}$,*

$$\|\hat{J}_\gamma - P_{\gamma,n}\|_{H^\infty(E_{\gamma,R})\to H^\infty(E_{\gamma,r})} \le c_{r,R}\,\frac{\cosh(n\log(r))}{\sinh(n\log(R))}, \tag{18}$$

$$\|P_{\gamma,n}\|_{H^\infty(E_{\gamma,R})\to H^\infty(E_{\gamma,r})} \le c_{r,R}\,\frac{\cosh(n\log(R)) + \cosh(n\log(r))}{\sinh(n\log(R))}, \tag{19}$$

where

$$c_{r,R} = \frac{\sinh(\log R)}{\cosh(\log(R)) - \cosh(\log(r))}.$$

3. Approximation of transfer operators with holomophic data

In this section we consider general transfer operators associated with holomorphic data on elliptic and annular domains. We show that Lagrange–Chebyshev and equidistant Lagrange approximants converge to the respective original operators at exponential speed in operator norm. These results are obtained using a factorisation argument already presented in various papers (see, for example, [BanJ1, BanJ2, SBJ1, BanJS]), together with Lemmas 2.1 and 2.9.

3.1. Transfer operators with holomorphic data on elliptic domains. For the remainder of this section we shall fix $1 < r < R$ and a pair of foci $\gamma = (\gamma_+, \gamma_-) \in \mathbb{C}^2$ with $\gamma_+ \neq \gamma_-$.

3.1. Definition. Let $1 < r < R$ and let $\mathcal{I}$ be a finite or countable index set. A *holomorphic map-weight system* is given by a family $(\Phi_i)_{i\in\mathcal{I}}$ of holomorphic maps in $H^\infty(E_{\gamma,R})$ satisfying

$$\bigcup_{i\in\mathcal{I}} \Phi_i(E_{\gamma,R}) \subseteq E_{\gamma,r}, \tag{20}$$

and a family of weights $(W_i)_{i\in\mathcal{I}}$ in $H^\infty(E_{\gamma,R})$ satisfying

$$S_{\gamma,R} = \sup\left\{\sum_{i\in\mathcal{I}} |W_i(z)| : z \in E_{\gamma,R}\right\} < \infty, \tag{21}$$

With each such holomorphic map-weight system we associate a *transfer operator* given by

$$\mathcal{L}f = \sum_{i\in\mathcal{I}} W_i \cdot f \circ \Phi_i. \tag{22}$$

In the following, we shall see that the transfer operator maps $H^\infty(E_{\gamma,r})$ compactly into itself; moreover, the transfer operator also maps $H^\infty(E_{\gamma,R})$ compactly into itself, and we will use the same symbol $\mathcal{L}$ for the operator on $H^\infty(E_{\gamma,r})$ as well as for its restriction to $H^\infty(E_{\gamma,R})$. We shall also see that the transfer operator on either $H^\infty(E_{\gamma,r})$ or $H^\infty(E_{\gamma,R})$ can be effectively approximated using Lagrange–Chebyshev interpolation, in a sense to be made precise below.

We start by proving that the transfer operator can be lifted to a bounded operator from $H^\infty(E_{\gamma,r})$ to $H^\infty(E_{\gamma,R})$.

3.2. Lemma. *The transfer operator arising from a holomorphic map-weight system maps $H^\infty(E_{\gamma,r})$ continuously to $H^\infty(E_{\gamma,R})$ with*

$$\|\mathcal{L}\|_{H^\infty(E_{\gamma,r})\to H^\infty(E_{\gamma,R})} \le S_{\gamma,R}.$$

Proof. Without loss of generality we shall assume that $\mathcal{I} = \mathbb{N}$. Fix $f \in H^\infty(E_{\gamma,r})$ and, for $k \in \mathbb{N}$, write $g_k(z) := \sum_{i=1}^k W_i(z) f(\Phi_i(z))$. We clearly have $g_k \in H^\infty(E_{\gamma,R})$. Since

$$|g_k(z)| \leq \sum_{i=1}^k |W_i(z)||f(\Phi_i(z))| \leq S_{\gamma,R} \|f\|_{H^\infty(E_{\gamma,r})} \tag{23}$$

for all $z \in E_{\gamma,R}$, it follows that the sequence $(g_k)_{k\in\mathbb{N}}$ is uniformly bounded on $E_{\gamma,R}$. Moreover, the limit $\lim_{k\to\infty} g_k(z) =: g(z)$ exists for every $z \in E_{\gamma,R}$. By Vitali's convergence theorem (see, for example, [N, Proposition 7]) the sequence $(g_k)_{k\in\mathbb{N}}$ thus converges uniformly on compact subsets of $E_{\gamma,R}$. Hence g is analytic on $E_{\gamma,R}$. Moreover, using (23) it follows that $|g(z)| \leq S_{\gamma,R}\|f\|_{H^\infty(E_{\gamma,r})}$ for any $z \in E_{\gamma,R}$. Thus $\mathcal{L}f = g \in H^\infty(E_{\gamma,R})$ and $\|\mathcal{L}f\|_{H^\infty(E_{\gamma,R})} \leq S_{\gamma,R} \|f\|_{H^\infty(E_{\gamma,r})}$, as required. □

The lemma above implies that the transfer operator of a holomorphic map-weight system is compact, when viewed as an operator from $H^\infty(E_{\gamma,r})$ into itself. In order to see this, let $\tilde{\mathcal{L}}\colon H^\infty(E_{\gamma,r}) \to H^\infty(E_{\gamma,R})$ denote the lifted transfer operator which is bounded by Lemma 3.2 and recall that the canonical embedding $\hat{J}_\gamma\colon H^\infty(E_{\gamma,R}) \to H^\infty(E_{\gamma,r})$ is compact by Lemma 2.9. Thus $\mathcal{L} = \hat{J}_\gamma\tilde{\mathcal{L}}$ is a compact endomorphism of $H^\infty(E_{\gamma,r})$. The same argument shows that $\mathcal{L} = \tilde{\mathcal{L}}\hat{J}_\gamma$ is a compact endomorphism of $H^\infty(E_{\gamma,R})$. This factorisation argument is also at the heart of the following theorem, our main result, which shows that $\mathcal{L}$ can be approximated at exponential speed using Lagrange–Chebyshev projections.

3.3. Theorem. *Let $\mathcal{L}$ be the transfer operator associated with a holomorphic map-weight system and $(P_{\gamma,n})_{n\in\mathbb{N}}$ the sequence of Lagrange–Chebyshev projections given in* (17). *Then the following holds.*

(i) $\mathcal{L}\colon H^\infty(E_{\gamma,r}) \to H^\infty(E_{\gamma,r})$ is compact and

$$\|\mathcal{L} - P_{\gamma,n}\mathcal{L}\|_{H^\infty(E_{\gamma,r})\to H^\infty(E_{\gamma,r})} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n \to \infty\,.$$

(ii) $\mathcal{L}\colon H^\infty(E_{\gamma,R}) \to H^\infty(E_{\gamma,R})$ is compact and

$$\|\mathcal{L} - \mathcal{L}P_{\gamma,n}\|_{H^\infty(E_{\gamma,R})\to H^\infty(E_{\gamma,R})} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n \to \infty\,.$$

Proof. Let $\tilde{\mathcal{L}}\colon H^\infty(E_{\gamma,r}) \to H^\infty(E_{\gamma,R})$ denote the lifted transfer operator from Lemma 3.2 and $\hat{J}_\gamma$ the canonical embedding operator from $H^\infty(E_{\gamma,R})$ to $H^\infty(E_{\gamma,r})$. For part (i) we use the factorisation $\mathcal{L} = \hat{J}_\gamma\tilde{\mathcal{L}}$. Lemmas 3.2 and 2.9 imply that $\mathcal{L}\colon H^\infty(E_{\gamma,r}) \to H^\infty(E_{\gamma,r})$ is compact; moreover

$$\begin{aligned}\|\mathcal{L} - P_{\gamma,n}\mathcal{L}\|_{H^\infty(E_{\gamma,r})\to H^\infty(E_{\gamma,r})} &= \left\|\hat{J}_\gamma\tilde{\mathcal{L}} - P_{\gamma,n}\tilde{\mathcal{L}}\right\|_{H^\infty(E_{\gamma,r})\to H^\infty(E_{\gamma,r})} \\ &\leq \left\|\hat{J}_\gamma - P_{\gamma,n}\right\|_{H^\infty(E_{\gamma,R})\to H^\infty(E_{\gamma,r})} \left\|\tilde{\mathcal{L}}\right\|_{H^\infty(E_{\gamma,r})\to H^\infty(E_{\gamma,R})}\end{aligned}$$

and the remaining assertion follows. For the proof of part (ii) write $\mathcal{L} = \tilde{\mathcal{L}}\hat{J}_\gamma$ and proceed as in part (i). □

3.4. Remark. A quick glance at the proof shows that the implied constants in the theorem above can be made explicit.

3.2. Transfer operators with holomorphic data on annular domains. In this subsection we shall state results analogous to those in the previous subsection, but now for transfer operators considered on the space of bounded holomorphic functions on annular domains containing the unit circle. The main application we have in mind is to generalised transfer operators associated with analytic expanding circle maps, that is, maps $\tau\colon \mathbb{T} \to \mathbb{T}$ with $\inf_{z\in\mathbb{T}} |\tau'(z)| > 1$. Note that any such map admits analytic extensions to annuli A_R for suitable $R > 1$.

3.5. Definition. Let $\tau\colon \mathbb{T} \to \mathbb{T}$ be an analytic expanding circle map. Suppose that there are positive real numbers r and R with $1 < r < R$ such that both τ and $1/\tau$ are holomorphic on A_R, and such that

$$\tau(A_r) \supset \mathrm{cl}(A_R),$$

where $\mathrm{cl}(\cdot)$ denotes the closure of a set of $\mathbb{C}$. To any such map τ and a weight function $w \in H^\infty(A_R)$ we associate the transfer operator $\mathcal{L}_\mathbb{T}\colon L^1(\mathbb{T}) \to L^1(\mathbb{T})$ defined by

$$(\mathcal{L}_\mathbb{T} f)(z) = \sum_{\zeta\colon \tau(\zeta)=z} \frac{w(\zeta)}{|\tau'(\zeta)|} f(\zeta), \tag{24}$$

where the summation extends over the (finitely many) pre-images of the point z under τ.

We shall now show that restricted to $H^\infty(A_R)$, this operator is compact and can be approximated at exponential speed using equidistant interpolation. In analogy with Lemma 3.2, we first show that $\mathcal{L}_\mathbb{T}$ lifts to a bounded operator from $H^\infty(A_r)$ to $H^\infty(A_R)$.

3.6. Lemma. *The operator $\mathcal{L}_\mathbb{T}$ given in Definition 3.5 maps $H^\infty(A_r)$ continuously to $H^\infty(A_R)$. In particular, we have*

$$\|\mathcal{L}_\mathbb{T}\|_{H^\infty(A_r)\to H^\infty(A_R)} \le \|w\|_{H^\infty(A_r)} \left(\frac{r^{\omega_\tau}}{\delta_+(r^{\omega_\tau})} + \frac{r^{-\omega_\tau}}{\delta_-(r^{-\omega_\tau})} \right),$$

where $\delta_+(r) = \min_{|z|=r} |\tau(z)| - R$ and $\delta_-(r) = 1/R - \max_{|z|=r} |\tau(z)|$, with $\omega_\tau = 1$ for orientation preseving and $\omega_\tau = -1$ for orientation reversing τ.

Proof. As τ is expanding, it is a K-fold covering for some $K > 1$, and $|\tau'| = \omega_\tau \tau'$. Let ϕ_i denote the i-th local inverse branch of τ, then $\mathcal{L}_\mathbb{T}$ can be written as

$$\mathcal{L}_\mathbb{T} f = \omega_\tau \sum_{i=1}^{K} (w \circ \phi_i) \cdot \phi_i' \cdot (f \circ \phi_i).$$

Given $f \in H^\infty(A_r)$ with $\|f\|_{H^\infty(A_r)} \le 1$, we want to show that $\mathcal{L}_\mathbb{T} f \in H^\infty(A_R)$. We follow the proof of [SBJ1, Lemma 2.3] by estimating the asymptotic behaviour of Fourier coefficients of $\mathcal{L}_\mathbb{T} f$. Using change of variables we can express the n-th Fourier coefficient of $\mathcal{L}_\mathbb{T} f$ as

$$c_n(\mathcal{L}_\mathbb{T} f) = \frac{\omega_\tau}{2\pi i} \int_\mathbb{T} \frac{w(z) f(z)}{\tau(z)^{n+1}}\, dz.$$

Fix $n \ge 0$. By [Rud, Theorem 17.11], the (nontangential) limit $f^*(z)$ for $z \in \partial A_r$ exists a.e. and f^* is integrable on ∂A_r. Moreover, as $z \mapsto \frac{w(z)}{\tau(z)^{n+1}}$ is holomorphic on $\mathrm{cl}(A_r)$ we may deform the contour to obtain

$$|c_n(\mathcal{L}_\mathbb{T} f)| = \left| \frac{\omega_\tau}{2\pi i} \int_{|z|=r^{\omega_\tau}} \frac{w(z) f^*(z)}{\tau(z)^{n+1}}\, dz \right| \le S_r \frac{r^{\omega_\tau}}{\left(\inf_{|z|=r^{\omega_\tau}} |\tau(z)|\right)^{n+1}} = S_r \frac{r^{\omega_\tau}}{(R + \delta_+(r^{\omega_\tau}))^{n+1}},$$

where $S_r = \|w\|_{H^\infty(A_r)}$. Similarly, for $n \ge 1$, we have

$$|c_{-n}(\mathcal{L}_\mathbb{T} f)| \le S_r r^{-\omega_\tau} \left(R^{-1} - \delta_-(r^{-\omega_\tau})\right)^{n-1}.$$

As τ is holomorphic on A_R, it follows by the Open Mapping Theorem that $\tau(A_r)$ is open. Thus the condition $\tau(A_r) \supset \mathrm{cl}(A_R)$ implies $\delta_+(r^{\omega_\tau}) > 0$ and $\delta_-(r^{-\omega_\tau}) > 0$. The same arguments as in the proof of [SBJ1, Lemma 2.3] now yield $\mathcal{L}_\mathbb{T} f \in H^\infty(A_R)$ as well as the claimed upper bound for the norm. □

Using the same factorisation argument already employed in Section 3.1, we obtain the following analogue of Theorem 3.3 for transfer operators on annular domains.

3.7. Theorem. *Let $\mathcal{L}_{\mathbb{T}}$ be the transfer operator given in Definition 3.5 and let $(Q_{2n})_{n\in\mathbb{N}}$ be the sequence of equidistant Lagrange interpolation projections given in* (7)*. Then the following holds.*

(i) $\mathcal{L}_{\mathbb{T}}\colon H^\infty(A_r)\to H^\infty(A_r)$ *is compact and*

$$\|\mathcal{L}_{\mathbb{T}} - Q_{2n}\mathcal{L}_{\mathbb{T}}\|_{H^\infty(A_r)\to H^\infty(A_r)} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n\to\infty.$$

(ii) $\mathcal{L}_{\mathbb{T}}\colon H^\infty(A_R)\to H^\infty(A_R)$ *is compact and*

$$\|\mathcal{L}_{\mathbb{T}} - \mathcal{L}_{\mathbb{T}}Q_{2n}\|_{H^\infty(A_R)\to H^\infty(A_R)} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n\to\infty.$$

3.8. Remark. As before, the implied constants in the theorem above can be made explicit.

3.3. Convergence of spectral data. Theorems 3.3 and 3.7 together with standard results from spectral perturbation theory now imply the desired convergence of spectral data of $P_{\gamma,n}\mathcal{L}$ to spectral data of $\mathcal{L}$, and similarly, the convergence of spectral data of $Q_{2n}\mathcal{L}_{\mathbb{T}}$ to spectral data of $\mathcal{L}_{\mathbb{T}}$. In order to avoid repetition, we shall state all results simultaneously for both Lagrangre–Chebyshev and equidistant Lagrange interpolation. For the remainder of this subsection, we write U_ρ, P_n, $\mathcal{L}$ for either $E_{\gamma,\rho}$, $P_{\gamma,n}$, $\mathcal{L}$ (as in Theorem 3.3) or A_ρ, Q_{2n}, $\mathcal{L}_{\mathbb{T}}$ (as in Theorem 3.7), respectively, where in both cases ρ may be either r or R.

3.9. Corollary. *Let $\mathcal{L}\colon H^\infty(U_r)\to H^\infty(U_r)$ denote the transfer operator as in Theorem 3.3 or Theorem 3.7 and let $\mathcal{L}_n = P_n\mathcal{L}$. Then the following holds.*

(i) Any convergent sequence $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n\in\operatorname{spec}(\mathcal{L}_n)$ converges to a spectral point of $\mathcal{L}$.

(ii) Conversely, for any $\mu\in\operatorname{spec}(\mathcal{L})$, there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n\in\operatorname{spec}(\mathcal{L}_n)$, such that $\mu_n\to\mu$ as $n\to\infty$. More precisely, if μ is an eigenvalue with ascent[1] ℓ, we have

$$|\mu-\mu_n| = O\left(\left(\frac{r}{R}\right)^{n/\ell}\right) \text{ as } n\to\infty.$$

(iii) Let $\mu\in\operatorname{spec}(\mathcal{L})\setminus\{0\}$ and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence with $\mu_n\in\operatorname{spec}(\mathcal{L}_n)$ such that $\mu_n\to\mu$ as $n\to\infty$. Writing $\mathcal{P}$ for the spectral projection associated with the eigenvalue μ of $\mathcal{L}$ and $(h_n)_{n\in\mathbb{N}}$ for a sequence of generalised eigenvectors associated with the eigenvalue μ_n of $\mathcal{L}_n$, normalised so that $\|h_n\|_{H^\infty(U_r)}=1$, we have

$$\|\mathcal{P}h_n - h_n\|_{H^\infty(U_r)} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n\to\infty.$$

Proof. Statements (i) and (ii) are known as Properties U and L, respectively, and follow from [ALL, Corollaries 2.7, 2.13], with the bound on the convergence rate following by combining our Theorem 3.3 (or Theorem 3.7, respectively) with Theorems 2.17, 2.18, and ensuing remarks in [ALL]. Part (i) of Theorem 3.3 (or Theorem 3.7, respectively) and [ALL, Proposition 2.9] finally yield statement (iii). □

3.10. Remark. Analogous spectral approximation results hold for $\mathcal{L}\colon H^\infty(U_R)\to H^\infty(U_R)$ and $\mathcal{L}_n = \mathcal{L}P_n$. However, these are less important from a practical perspective. The reason for this is that, while the non-zero eigenvalues of $P_n\mathcal{L}$ and $\mathcal{L}P_n$ coincide, this is not the case for the corresponding generalised eigenspaces, and, as we shall see in Section 4, the generalised eigenvectors of $P_n\mathcal{L}$ are easier to calculate than those of $\mathcal{L}P_n$.

The corollary above provides estimates for the speed of convergence of the generalised eigenvectors of $P_n\mathcal{L}$ to the corresponding generalised eigenspace of $\mathcal{L}$. Similar results can be obtained for the corresponding eigenfunctionals. In order to see this, let $H^\infty(U_R)^*$ denote the dual space of $H^\infty(U_R)$ equipped with the usual strong dual topology turning $H^\infty(U_R)^*$ into a Banach space.

[1] An eigenvalue μ of an operator T is said to have *ascent* ℓ, if ℓ is the smallest integer such that the kernel of $(\mu I - T)^\ell$ equals that of $(\mu I - T)^{\ell+1}$. In particular, if μ is algebraically simple, then $\ell = 1$.

Part (ii) of Theorem 3.3 (or Theorem 3.7, respectively) implies that

$$\|\mathcal{L}^* - P_n^*\mathcal{L}^*\|_{H^\infty(U_R)^*\to H^\infty(U_R)^*} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n\to\infty,$$

that is, the adjoint approximant $P_n^*\mathcal{L}^*$ converges to the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ in the operator norm on $H^\infty(U_R)^*$, which in turn implies convergence of the corresponding generalised eigenspaces. More precisely, we have the following dual analogue of part (iii) of Corollary 3.9.

3.11. Corollary. *Let $\mathcal{L}^*\colon H^\infty(U_R)^* \to H^\infty(U_R)^*$ denote the adjoint transfer operator as in Theorem 3.3 or Theorem 3.7, and let $\mathcal{L}_n^* = P_n^*\mathcal{L}^*$. Suppose that $\mu \in \operatorname{spec}(\mathcal{L}^*)\setminus\{0\}$ and that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence with $\mu_n \in \operatorname{spec}(\mathcal{L}_n^*)$ such that $\mu_n \to \mu$ as $n\to\infty$. Writing $\mathcal{P}^*$ for the spectral projection associated with the eigenvalue μ of $\mathcal{L}^*$ and $(h_n^*)_{n\in\mathbb{N}}$ for a sequence of generalised eigenvectors associated with the eigenvalue μ_n of $\mathcal{L}_n^*$, normalised so that $\|h_n^*\|_{H^\infty(U_R)^*} = 1$, we have*

$$\|\mathcal{P}^*h_n^* - h_n^*\|_{H^\infty(U_R)^*} = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n\to\infty.$$

4. Applications

In this section we apply the Lagrange approximation algorithm to several approximation problems involving transfer operators and compare its performance with algorithms available in the literature.

4.1. Spectral data of Lagrange approximants. Before going into details we briefly review how to calculate eigendata of finite-rank approximants obtained by applying Lagrange projections to the transfer operator of a holomorphic map-weight system. In order to avoid cluttered notation, we write X for any of the underlying Banach spaces discussed in the previous section, $L\colon X\to X$ for the transfer operator arising from a holomorphic map-weight system, which we merely need to assume bounded for the purpose of this discussion, and $P\colon X\to X$ for a bounded projection of rank n, given by

$$Pf = \sum_{l=1}^n e_l^*(f)e_l \qquad (f\in X),$$

where $e_l \in X$ and $e_l^* \in X^*$ for $l = 1,\dots,n$. Here P may be thought of as any of the Lagrange projections given in Equations (5), (16) or (17). Given L and P as above, define an $n\times n$ matrix M by

$$M_{kl} = e_k^*(Le_l) \qquad (k,l\in\{1,\dots,n\}). \tag{25}$$

It is not difficult to see, for example, by appealing to the Principle of Related Operators (see [P, Section 3.3]) that the operators PL, LP, PLP and M have the same non-zero eigenvalues with the same algebraic multiplicities; in particular,

$$\operatorname{spec}(PL)\setminus\{0\} = \operatorname{spec}(LP)\setminus\{0\} = \operatorname{spec}(PLP)\setminus\{0\} = \operatorname{spec}(M)\setminus\{0\}.$$

As for the corresponding generalised eigenvectors and eigenfunctionals, a short calculation shows that for any non-zero $\mu\in\mathbb{C}$ and any $k\in\mathbb{N}$ we have

$$h\in\ker((\mu I - PLP)^k) \text{ iff } h = \sum_{l=1}^n x_l e_l \text{ with } x\in\ker((\mu I - M)^k), \tag{26}$$

$$h^*\in\ker((\mu I - P^*L^*P^*)^k) \text{ iff } h^* = \sum_{l=1}^n x_l e_l^* \text{ with } x\in\ker((\mu I - M^T)^k). \tag{27}$$

Thus the generalised eigenvectors and eigenfunctionals of PLP can be obtained from the generalised right and left eigenvectors of M. Finally, we note that for any non-zero $\mu\in\mathbb{C}$ and any $k\in\mathbb{N}$ we have

$$\ker((\mu I - PLP)^k) = \ker((\mu I - PL)^k), \tag{28}$$

so that the spectral convergence results contained in Corollaries 3.9 and 3.11 can be applied to the approximate eigendata obtained through (26) and (27).

The proof of (28) is straightforward: given $h \in \ker(\mu I - PLP)$ we have $\mu h = PLPh$, so h must be in the range of P, hence $Ph = h$, from which $\mu h = PLh$, that is, $h \in \ker(\mu I - PL)$. For the converse, suppose that $h \in \ker(\mu I - PL)$, so h must again be in the range of P, from which $Ph = h$, and so $h \in \ker(\mu I - PLP)$. The proof of the general case with $k > 1$ is similar.

For the sake of completeness, we present a simple algorithm for the computation of matrix elements M_{kl} in (25) for L a transfer operator associated with a map-weight system holomorphic on standard ellipses, that is $\gamma = (-1, 1)$, and P the Lagrange–Chebyshev interpolation operator in (16).

Algorithm 1 Lagrange–Chebyshev approximation of a transfer operator

Require: n, $\{W_j\}_{j=0,\dots,d-1}, \{\Phi_j\}_{j=0,\dots,d-1}$
  {Evaluation of data at interpolation points}
  for $m = 0$ to $n-1$ **do**
    $x_m \leftarrow \cos \frac{\pi(2m+1)}{2n}$
    for $j = 0$ to $d-1$ **do**
      $w_{jm} \leftarrow W_j(x_m)$
      $\phi_{jm} \leftarrow \Phi_j(x_m)$
    end for
  end for
  {Computation of matrix entries}
  for $k = 0$ to $n-1$ **do**
    for $l = 0$ to $n-1$ **do**
      $M_{kl} \leftarrow \frac{2-\delta_{0,k}}{n} \sum_{m=0}^{n-1} T_k(x_m) \sum_{j=0}^{d-1} w_{jm} T_l(\phi_{jm})$
    end for
  end for
  return M

4.1. Remark. It it not difficult to see that $M = AB$ with $A_{kl} = \frac{2-\delta_{0,k}}{n} T_k(x_l)$ and $B_{kl} = (LT_l)(x_k)$ for $k, l = 0, \dots, n-1$, which yields a more efficient way to compute M in the algorithm above.

4.2. Decay of correlations. In [BahB], Bahsoun and Bose consider discretisation schemes based on piecewise linear approximations to given functions and apply them to transfer operators arising from interval maps with the transfer operator acting on the Banach space of Lipschitz continuous functions. The resulting scheme is shown to provide convergent approximations to the invariant densities together with rigorous error bounds in a topology stronger than pointwise convergence (that is, stronger than the usual L^1-convergence obtained through the standard Ulam method).

As an example, the authors establish rigorous error bounds for their approximation scheme when applied to the full-branch map $T\colon [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} \frac{11x}{1-x} & 0 \le x < \frac{1}{12}, \\ 12x - i & \frac{i}{12} < x \le \frac{i+1}{12}, \end{cases}$$

where $i = 1, \dots, 11$. Using the notation from Section 3.1 we obtain a holomorphic map-weight system on an ellipse $E_{\gamma,R} \supset [0,1]$ with $\gamma = (0,1)$ and suitable $R > 1$ (in fact, any $R \in (10, 20)$ is suitable) given by a family $\{\Phi_i\}_{i=0}^{11}$ with $\Phi_0(x) = \frac{x}{11+x}$ and $\Phi_i(x) = \frac{x+i}{12}$ for $i = 1, \dots, 11$, and a family of weights $\{W_i\}_{i=0}^{11}$ given by $W_i(x) = \Phi_i'(x)$. The associated transfer operator given by (22) is well-defined and compact on $H^\infty(E_{\gamma,R})$ with a simple leading eigenvalue at 1. As is known, the rate of correlation decay is determined by the subleading eigenvalue of $\mathcal{L}$, which can easily be approximated using the Lagrange–Chebyshev approximation algorithm.

In [BahB], the subleading eigenvalue of λ_2 of $\mathcal{L}$ was numerically observed to be simple, which by Corollary 3.9(ii) implies

$$|\lambda_2 - \lambda_{2,n}| = O\left(\left(\frac{r}{R}\right)^n\right) \text{ as } n \to \infty,$$

where $\lambda_{2,n}$ is the simple subleading eigenvalue of $\mathcal{L}_n$ (for sufficiently large n). Below we compute $\lambda_{2,n}$ for different values of n illustrating the rapid exponential convergence of the scheme.

2	0.0899609091606775605271181343464894043253988825186233560706
12	0.0900761270052955777611934889786172935065120132357444867624
22	0.0900761270052955778472464929999485481037667345626173002032
32	0.0900761270052955778472464929999485626943805990355232965894
42	0.0900761270052955778472464929999485626943805990355246068579

In order to obtain a good contraction ratio, we performed a numerical search in the collection of ellipses $E_{\gamma,R}$ with $\gamma = (0,1)$ and $R \in (1.01, 40)$. For each such ellipse, we numerically determined the smallest ellipse $E_{\gamma,r}$ containing $\bigcup_i \Phi_i(E_{\gamma,R})$. The best contraction ratio found in this way turned out to be $(r/R) \approx 0.225$, which occurred for $R \approx 16.99$. Comparing the resulting upper bound with the empirically observed convergence rate $|\lambda_{2,n} - \lambda_{2,n-1}|$, see Figure 1, shows that the theoretical bound is rather conservative in this case.

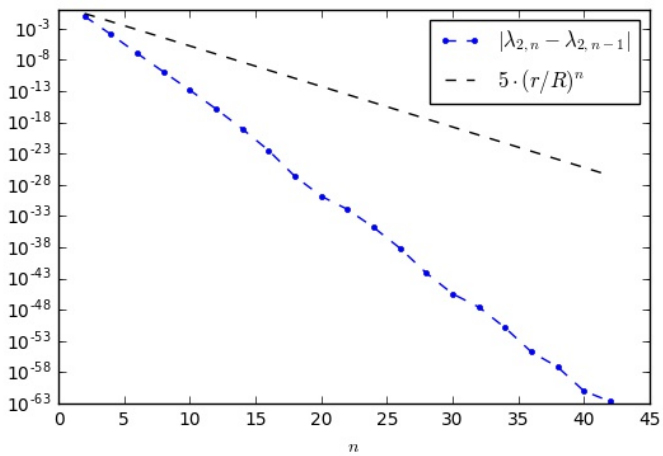

FIGURE 1. Absolute difference of subsequent approximations $\lambda_{2,n}$ as a function of n, computed using the Lagrange–Chebyshev algorithm (Algorithm 1), compared to the upper bound given by the optimal contraction ratio in a family of confocal ellipses with foci $\gamma = (0,1)$. The scaling constant 5 for the theoretical bound is chosen for visual reasons for both curves to intersect at $n = 2$.

4.3. Eigenvalues of transfer operators for circle maps. In this subsection we shall approximate the spectrum of transfer operators associated with expanding circle maps. We shall choose Blaschke products as benchmark maps, as the corresponding spectrum is available explicitly (see [SBJ1, BanJS]). Using the same notation as in Section 3.2, let τ be a Blaschke product of degree two given by

$$\tau(z) = \left(\frac{z-\mu}{1-\bar{\mu}z}\right)^2, \quad |\mu| < 1.$$

For $|\mu| < 1/3$, the map τ yields an expanding circle map. We shall approximate the spectrum of the transfer operator $\mathcal{L}_{\mathbb{T}}$ in (24) with weight function given by[2] $w(z) = 1/\tau'(z)$. Using results from

[2]The leading eigenfunctional of this transfer operator yields the measure of maximal entropy for τ.

[BanJS], it is not difficult to see that the spectrum of $\mathcal{L}_{\mathbb{T}}$ on $H^\infty(A_R)$ for a suitable annulus A_R is given by

$$\operatorname{spec}(\mathcal{L}_{\mathbb{T}}) = \{0,2\} \cup \{(\tau'(z_0))^j, \overline{(\tau'(z_0))^j}\}_{j\in\mathbb{N}},$$

where z_0 is the unique attracting fixed point of τ in the open unit disk $\mathbb{D}$.

For our numerical experiments we use $\mu = 0.33\exp(i\pi/2)$ and seek to approximate one of the subleading eigenvalues. Ordering the eigenvalues by decreasing modulus

$$\lambda_1 = 2,\ \lambda_2 = \tau'(z_0),\ \lambda_3 = \overline{\tau'(z_0)},\ \lambda_4 = (\tau'(z_0))^2,\ \lambda_5 = \overline{(\tau'(z_0))}^2, \dots,$$

we shall focus on the 7-th eigenvalue in this sequence. The fixed point z_0 of τ as well as $\tau'(z_0)$ are available explicitly, however, as the expression is rather cumbersome, we shall only give the numerical value of λ_7 using the first 60 digits

$$\begin{aligned}\Re(\lambda_7) &= 0.092670812973910200449109943460780953297549127162781618050227\dots\\ \Im(\lambda_7) &= -0.142165954484616119517212417389833629637614779964273017031665\dots.\end{aligned}$$

Employing the equidistant Lagrange interpolation algorithm to calculate matrix representations of $Q_{2n}\mathcal{L}_{\mathbb{T}}Q_{2n}$ we obtain the following approximations of the 7-th eigenvalue of $\mathcal{L}_{\mathbb{T}}$ for increasing values of n.

n	$\Re(\lambda_{7,n})$	$-\Im(\lambda_{7,n})$
23	0.0926708129739099154284829115157938122368669 3	0.1421659544846163819762201567962912837491514
33	0.0926708129739102004491105145187362515442754 8	0.1421659544846161195172136940142097007716003
43	0.0926708129739102004491099434607809397778141 0	0.1421659544846161195172124173898336025839907
53	0.0926708129739102004491099434607809532975491 0	0.1421659544846161195172124173898336296376148

Inspection of the table above shows that, for $n = 53$, the first 42 decimal places of the approximate eigenvalue $\lambda_{7,n}$ coincide with the theoretical value λ_7. As a result, we see that subleading complex eigenvalues close to 0 of general transfer operators can be approximated effectively using the equidistant Lagrange interpolation algorithm.

4.4. Lyapunov exponents of random matrix products. In this subsection we shall illustrate the use of the Lagrange–Chebyshev algorithm to approximate Lyapunov exponents of random products of matrices. This is a rather challenging task, in general. However, if all matrices are positive, it is possible to use thermodynamic formalism to obtain the top Lyapunov exponent from a certain family of transfer operators through periodic orbit expansions. This approach is originally due to Pollicott [Po]; more recently, an effective version has been proposed by Jurga and Morris [JM]. In the following we shall show how, instead of periodic orbit expansions, the Lagrange–Chebyshev algorithm can be used effectively in this setup.

Let $\mathcal{A} = \{A_1, \dots, A_K\}$ be a finite set of positive invertible 2×2 matrices, let $(p_1, \dots, p_K)$ be a probability vector and denote by $\mathbb{P}_p$ the associated Bernoulli measure on the space of sequences $\Omega = \{1, \dots, K\}^{\mathbb{N}}$. The (top) Lyapunov exponent of $\mathbb{P}_p$ is given by

$$\Lambda = \Lambda(\mathcal{A}, p) = \lim_{n\to\infty} \frac{1}{n} \int \log \|A_{\omega_1} \cdots A_{\omega_n}\|\, d\mathbb{P}_p(\omega),$$

where $\omega \in \Omega$, and $\|\cdot\|$ denotes any matrix norm. Moreover, by [FK] for $\mathbb{P}_p$-a.e. $\omega = (\omega_n)_{n\in\mathbb{N}}$ we have

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \log \|A_{\omega_1} \cdots A_{\omega_n}\|.$$

As was shown in [Po] and [JM], the Lyapunov exponent Λ can be expressed as the derivative of the top eigenvalue of $\mathcal{L}_t$ with respect to t at $t = 0$, where $\{\mathcal{L}_t\}_{t\in\mathbb{C}}$ is a certain one-parameter family of transfer operators obtained as follows. To each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}$$

we associate a Moebius map ϕ_A given by

$$\phi_A(z) = \frac{(a-b)z+b}{w_A(z)}$$

with $w_A(z) = (a+c-b-d)z+b+d$. Writing $D_\rho(z_0)$ for a disk in $\mathbb{C}$ with radius ρ and centre z_0, it is not difficult to see that ϕ_A is a holomorphic self map of $D_{\frac{1}{2}}(\frac{1}{2})$ with $\phi_A(D_{\frac{1}{2}}(\frac{1}{2})) \subset D_\rho(\frac{1}{2})$ for suitable $\rho < 1/2$. Moreover, one can show that $\phi_A(E_{\gamma,R}) \subset E_{\gamma,r}$ for a suitable $1 < r < R$ and $\gamma = (\gamma_+, \gamma_-)$, with γ_+, γ_- small (distinct) perturbations of $1/2$. We can now define a (so-called annealed) transfer operator $\mathcal{L}$ on $H^\infty(E_{\gamma,R})$ associated to $\mathcal{A}$ and p by

$$\mathcal{L}_0 f = \sum_{i=1}^{K} p_i f \circ \phi_{A_i}.$$

This operator is compact with leading eigenvalue $\lambda_0 = 1$, which turns out to be simple [JM, Proposition 2.3] and the corresponding eigenfunction is the constant function $h_0 = \mathbf{1}$. We now define a family of perturbed transfer operators

$$\mathcal{L}_t f = \sum_{i=1}^{K} p_i (w_{A_i})^t f \circ \phi_{A_i} \qquad (t \in \mathbb{C}),$$

which are easily seen to be well-defined operators on $H^\infty(E_{\gamma,R})$. This follows by observing that each weight function $(w_{A_i})^t = \exp(t \log w_{A_i})$, where log denotes the principal branch of the complex logarithm, is a bounded holomorphic function on $E_{\gamma,R}$ as $\Re(w_{A_i}(z)) > 0$ for $z \in E_{\gamma,R}$. Since $t \mapsto \mathcal{L}_t$ is a holomorphic family in t and λ_0 is an algebraically simple isolated eigenvalue of $\mathcal{L}_0$, it follows by standard analytic perturbation theory (see, for example, [K, Chapter II, § 1.8]) that the largest (in modulus) eigenvalue λ_t of $\mathcal{L}_t$ as well as the corresponding spectral projection $\mathcal{P}_t$ are holomorphic in t on an open neighbourhood around $t = 0$. We write $\mathcal{P}_t f = h_t^*(f) h_t$, where $h_t \in H^\infty(E_{\gamma,R})$ is the eigenvector of $\mathcal{L}_t$ corresponding to λ_t and $h_t^* \in H^\infty(E_{\gamma,R})^*$ the corresponding eigenfunctional, normalised so that $h_t^*(\mathbf{1}) = 1$. By [JM, Proposition 3.1], it follows that the Lyapunov exponent can be expressed as the derivative of the top eigenvalue λ_t at $t = 0$, that is,

$$\Lambda = \frac{\partial \lambda_t}{\partial t}\Big|_{t=0}.$$

As $\mathcal{L}_t$, h_t and h_t^* are holomorphic families in t on a neighbourhood around $t = 0$, a standard computation (see, for example, [K, Chapter II, § 2.2, Remark 2.2]), shows that

$$\Lambda = h_0^*(\mathcal{M}_0 h_0),$$

where $\mathcal{M}_0$ denotes the derivative of $\mathcal{L}_t$ at $t = 0$, which is given by

$$\mathcal{M}_0 f = \sum_{i=1}^{K} p_i \log(w_{A_i}) f \circ \phi_{A_i}.$$

The operator $\mathcal{M}_0$ is again a well-defined operator on $H^\infty(E_{\gamma,R})$, by the same arguments as before.

We shall now approximate the Lyapunov exponent $\Lambda = h_0^*(\mathcal{M}_0 h_0)$ using Lagrange–Chebyshev approximation. For $n \in \mathbb{N}$ let $\lambda_{0,n}$ denote the leading eigenvalue of $\mathcal{L}_{0,n} = \mathcal{L}_0 P_{\gamma,n}$ and $h_{0,n}$ the corresponding leading eigenfunction. Using (discrete) orthogonality properties of Chebyshev polynomials it is not difficult to see that $\lambda_{0,n} = \lambda_0 = 1$ with $h_{0,n} = h_0 = \mathbf{1}$. The eigenfunctional $h_{0,n}^*$ of $\mathcal{L}_{0,n}$ corresponding to $\lambda_{0,n} = 1$, normalised so that $h_{0,n}^*(\mathbf{1}) = 1$, can be obtained numerically, as described in the opening paragraphs of this section, and yields an approximation

$$\Lambda_n = h_{0,n}^*(\mathcal{M}_0 \mathbf{1})$$

for the Lyapunov exponent Λ. In order to estimate the speed of convergence, we note that

$$\mathcal{P}_0^* h_{0,n}^* = h_0^*,$$

where $\mathcal{P}_0$ denotes the adjoint of the spectral projection $\mathcal{P}_0$ of $\mathcal{L}_0$ associated with $\lambda_0 = 1$. In order to see this observe that for any $f \in H^\infty(E_{\gamma,R})$ we have

$$\mathcal{P}_0^* h_{0,n}^*(f) = h_{0,n}^*(\mathcal{P}_0 f) = h_0^*(f) h_{0,n}^*(\mathbf{1}) = h_0^*(f).$$

Thus, we have

$$\begin{aligned}|\Lambda - \Lambda_n| = |h_0^*(\mathcal{M}_0\mathbf{1}) - h_{0,n}^*(\mathcal{M}_0\mathbf{1})| &= |(\mathcal{P}_0^* h_{0,n}^* - h_{0,n}^*)(\mathcal{M}_0\mathbf{1})| \\ &\leq \|\mathcal{P}_0^* h_{0,n}^* - h_{0,n}^*\|_{H^\infty(E_{\gamma,R})^*} \|\mathcal{M}_0\mathbf{1}\|_{H^\infty(E_{\gamma,R})},\end{aligned}$$

and so, by Corollary 3.11,

$$|\Lambda - \Lambda_n| = O\left(\left(\frac{r}{R}\right)^n\right).$$

Thus, the approximations converge exponentially in n, the size of the matrix representing the Lagrange–Chebyshev approximant, with the speed determined by the parameters of the ellipses $E_{\gamma,R}$ and $E_{\gamma,r}$ satisfying

$$\bigcup_i \phi_{A_i}(E_{\gamma,R}) \subset E_{\gamma,r},$$

that is, complex contraction properties of the maps ϕ_{A_i}.

We shall now test the performance of our algorithm using examples from [JM].

(1) In Example 5.1 of [JM], the matrices and probability vector are chosen to be

$$\mathcal{A} = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \right\} \text{ and } p = (\frac{1}{2}, \frac{1}{2}).$$

In this setting, the matrices in $\mathcal{A}$ strongly contract the positive quadrant, resulting in a highly effective approximation involving periodic orbits up to order 9 justifiably accurate to 31 decimal places. Using our Lagrange–Chebyshev algorithm with $n = 65$, we obtain the same reported value of

$$\Lambda_n = 1.1433110351029492458432518536555882994025.$$

(2) In Example 5.2 of [JM], the following choices are made

$$\mathcal{A} = \left\{ \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \right\} \text{ and } p = (\frac{1}{2}, \frac{1}{2}).$$

Note that in this case, we have $a_i + c_i = b_i + d_i$ so the weights w_{A_i} are constant functions for $i = 1, 2$ hence $\mathcal{M}_0\mathbf{1} = \mathbf{1}$, and therefore $\Lambda = \sum_{i=1}^K p_i \log(b_i + d_i)$, which is the first entry of the matrix representation of $\mathcal{L}_n$ for any n. In this case, the Lagrange–Chebyshev algorithm immediately yields the correct value, whereas the periodic orbit method from [JM] only converges moderately fast.

(3) Perturbing the matrix entries of the previous trivial example, we now choose

$$\mathcal{A} = \left\{ \begin{pmatrix} 3.1 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 5.1 & 2 \\ 2 & 5 \end{pmatrix} \right\} \text{ and } p = (\frac{1}{2}, \frac{1}{2}).$$

Applying the Lagrange–Chebyshev algorithm, we obtain very fast convergence in n, as seen below

1	1.67587224897131252134767223196558918068328794582311789987 98
10	1.676050187659018330526782391760452984608853288600186187434 2
20	1.676050187659018330529800187539023451047382520427598134846 7
30	1.676050187659018330529800187539023451047313771360106564229 8
40	1.676050187659018330529800187539023451047313771360106564229 8

In this case, the assumption $\Re(w_{A_i}(z)) > 0$ for $i = 1, 2$ is satisfied for z with $\Re(z) \geq -40$, which in particular guarantees that the weights $(w_{A_i})^t$ are holomorphic on $E_{\gamma,R}$ for $\gamma = (0, 1)$ and $1 < R < \exp(\operatorname{arccosh}(40)) \approx 80$. Numerically, we may also verify that this choice of

ellipses satisfies $\bigcup_i \phi_{A_i}(E_{\gamma,R}) \subset E_{\gamma,r}$ with $r < R$, yielding an optimal contraction ratio of $(r/R) \approx 0.53$ for $R \approx 9.53$.

4.5. Approximation of stationary probability measures for iterated function systems. In [CJ], Cipriano and Jurga study approximations of integrals with respect to stationary probability measures associated to iterated function systems on the interval, using an idea going back to Jenkinson and Pollicott [JP3]. Their setting is an iterated function system $\{\Phi_i\}_{i=1}^K$ consisting of Lipschitz contractions $\Phi_i\colon [0,1] \to [0,1]$. Given a probability vector $p = (p_1, \dots, p_K)$, there exists a unique probability measure ν such that

$$\int f\, d\nu = \sum_{i=1}^K p_i \int_{[0,1]} f \circ \Phi_i\, d\nu$$

for every continuous function $f\colon [0,1] \to \mathbb{R}$, see [Hu]. Assuming additionally that each Φ_i extends holomorphically to a neighbourhood of $[0,1]$ and satisfies assumption (20) for a suitable ellipse $E_{\gamma,R} \supset [0,1]$, we shall again consider the (annealed) transfer operator

$$\mathcal{L}f = \sum_{i=1}^K p_i (f \circ \Phi_i),$$

which is a well-defined and compact operator on $H^\infty(E_{\gamma,R})$. It has a simple eigenvalue 1 with eigenfunction $\mathbf{1}$, and the stationary measure ν turns out to be the eigenfunctional h^* of $\mathcal{L}$ corresponding to the eigenvalue 1, normalised so that $h^*(\mathbf{1}) = 1$, that is

$$h^*(f) = \int_{[0,1]} f\, d\nu.$$

Using the Lagrange–Chebyshev algorithm, we can obtain effective approximations of the stationary measure through eigenfunctionals of the approximants $\mathcal{L}_n = \mathcal{L}P_{\gamma,n}$. In order to see this, note that 1 is a simple eigenvalue of $\mathcal{L}_n$ and $h_n = \mathbf{1}$ is the corresponding eigenfunction of $\mathcal{L}_n$. The corresponding eigenfunctional h_n^* can be obtained from the matrix representation of $\mathcal{L}_n$, as explained in the introduction to this section.

In [CJ], the authors consider various integrals with respect to the stationary measure arising from different iterated function systems, including an application to the calculation of Lyapunov exponents. Here, the Lyapunov exponent of the iterated function system (Φ, p) with respect to the stationary measure ν is given by

$$\Lambda = -\int_{[0,1]} \sum_{i=1}^K p_i \log|\Phi_i'(x)|\, d\nu(x).$$

Suppose now that there is an r with $1 < r < R$ such that

$$\bigcup_i \Phi_i(E_{\gamma,R}) \subset E_{\gamma,r}.$$

Suppose also that the function $x \mapsto \sum_i p_i \log|\Phi_i'(x)|$ has a holomorphic extension g, which is bounded and holomorphic on the elliptic domain $E_{\gamma,R}$. Using the approximate eigenfunctional h_n^* given above, we obtain an approximation Λ_n to the Lyapunov exponent Λ by setting

$$\Lambda_n = h_n^*(g).$$

In order to estimate the speed of convergence of this approximation we proceed as in the previous subsection. Let $\mathcal{P}$ denote the spectral projection of $\mathcal{L}$ associated with the leading simple eigenvalue 1, that is

$$\mathcal{P}f = h^*(f)\mathbf{1}.$$

As before, we have $\mathcal{P}^* h_n^* = h^*$, hence

$$|\Lambda - \Lambda_n| = |h^*(g) - h_n^*(g)| = |\mathcal{P}^* h_n^*(g) - h_n^*(g)| \le \|\mathcal{P}^* h_n^* - h_n^*\|_{H^\infty(E_{\gamma,R})^*} \|g\|_{H^\infty(E_{\gamma,R})},$$

and Corollary 3.11 now yields

$$|\Lambda - \Lambda_n| = O\left(\left(\frac{r}{R}\right)^n\right).$$

We shall now compare our results to Example 6.8 in [CJ]. Let

$$\Phi_1(x) = \frac{1}{6}\sin(\pi x/4) + \frac{1}{4}, \qquad \Phi_2(x) = \frac{1}{3}\sin(\pi x/4) + \frac{2}{3}, \qquad p = (\frac{1}{3}, \frac{2}{3}).$$

We may now approximate the corresponding Lyapunov exponent by Λ_n for sufficiently large n, for example by

$$\Lambda_{100} = \quad 1.736720814737319877193356690960513773360205906006376079918873624791932498455557168841091,$$

where the first 89 digits coincide with the value reported in [CJ], which was computed using multipliers of periodic orbits up to period 18. The optimal contraction ratio of ellipses was numerically found to be $(r/R) \approx 0.4138$ in this case.

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