ORTHONORMAL EXPANSIONS OF INVARIANT DENSITIES FOR EXPANDING MAPS

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Abstract. We give a novel way of constructing the density function for the absolutely continuous invariant measure of piecewise expanding C^ω Markov maps. This is a classical problem, with one of the standard approaches being Ulam's method [U] of phase space discretisation.

Our method hinges instead on the expansion of the density function with respect to an L^2 orthonormal basis, and the computation of the expansion coefficients in terms of the periodic orbits of the expanding map. The efficiency of the method, and its extension to C^k expanding maps, are also discussed.

1. Introduction

Let T be either a C^ω (i.e. real analytic) expanding map of the circle, or a piecewise C^ω expanding Markov map (see §2 for the definition) of the interval. It is well known that there exists a T-invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure (cf. [LY]). If T is topologically mixing[1] then this absolutely continuous invariant measure is unique and ergodic, with strictly positive C^ω density function ϱ.

A central problem in ergodic theory is the computation of time averages of the form $\lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1} g(T^r x)$, for integrable functions g. The straightforward evaluation of $\frac{1}{N}\sum_{r=0}^{N-1} g(T^r x)$ for large N is known to be an inefficient way of approximating the limit (cf. [ER]). An application of Birkhoff's ergodic theorem reduces the problem, for Lebesgue almost every x, to a computation of the space average $\int g\, d\mu = \int g(x)\varrho(x)dx$. However for general nonlinear expanding maps the density ϱ is not explicitly known (i.e. is not available in closed form), so that an approximation scheme for evaluating $\int g\, d\mu$ is required.

In [JP] we gave an efficient method for computing such integrals, provided the function g is real analytic. The periodic points of period up to M are used to construct a sequence of approximations to $\int g\, d\mu$, whose rate of convergence to $\int g\, d\mu$ is *superexponential in* M. Thus we have a very rapidly convergent algorithm for computing the time average of a given (real analytic) function g.

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[1] Mixing is automatic if T is an expanding circle map.

If, however, we want to compute time averages for *many* functions g, the required high number of iterations of our algorithm may outweigh its inherent efficiency. In this case it would be preferable to accurately approximate, once and for all, the density function ϱ. Thereafter, for all functions g, which now may be chosen as merely L^1 rather than C^ω, the integral $\int g\, d\mu = \int g(x)\varrho(x)dx$ can be approximated *directly*, without recourse to an algorithm. That is, each integral may be evaluated approximately by either a symbolic or a numerical method, and the time required for such evaluations is comparatively very small. The classical method for approximating density functions, first proposed by Ulam [U], is to discretise the system by partitioning the phase space. The density is then approximated by the dominant eigenvector of a finite matrix. Recent work on Ulam's method includes [DS], [Froy], [Hunt], [KMY], [Liv], [Mu].

In the present paper we will describe how to perform a one-off computation of ϱ using periodic points, and give error estimates in terms of the largest period used. The starting point is to view the analytic function ϱ as an element of a suitable L^2 space, and expand $\varrho = \sum_n (\varrho, f_n) f_n$ in terms of an orthonormal basis $\{f_n\}$. If the interval is parametrised as $I = [-1, 1]$, then a natural expansion is in terms of normalised *Legendre polynomials* $f_n = (n + \frac{1}{2})^{1/2} P_n$, for $n \geq 0$, where P_n is the n^{th} Legendre polynomial (see §3 for the definition). In this case

$$(\varrho, f_n) = \int_{-1}^{1} \varrho(x)\overline{f_n(x)}\, dx = \int \overline{f_n}\, d\mu$$

is the inner product in the Hilbert space $L^2(I, dx)$.

If T is a map of the circle then the density ϱ will be analytic, and hence L^2, on the circle. Equivalently, ϱ can be considered as a continuous function on the interval[2] $I = [-1, 1]$ admitting an extension as a function of period two on the real line. A natural complete orthonormal set for $L^2(\mathbb{T}, dx)$ consists of Fourier characters $f_n(x) = 2^{-1/2} e^{\pi n i x}$, and we have the *Fourier series* expansion

$$\varrho = \sum_{n=-\infty}^{\infty} (\varrho, f_n) f_n\,,$$

where the Fourier coefficients (ϱ, f_n) are given by

$$(\varrho, f_n) = 2^{-1/2} \int_{-1}^{1} \varrho(x) e^{-\pi n i x}\, dx = 2^{-1/2} \int e^{-\pi n i x}\, d\mu(x)\,.$$

For our purposes the key feature of L^2 space is that the inner product is defined in terms of *integration*. In particular the expansion coefficients (ϱ, f_n) are given by integrating complex conjugates of basis functions with respect to the invariant measure μ. If we require that every conjugate basis element $\overline{f_n}$ be real analytic, then each coefficient (ϱ, f_n) can be approximated by means of the periodic point algorithm of [JP]. That is, there is a

[2]For consistency of notation we retain the parametrisation $I = [-1, 1]$. It is of course more common to use $I = [0, 1]$, in which case the Fourier basis is $f_n(x) = e^{2\pi i n x}$. See also §4.

sequence $(\varrho, f_n)^{(M)}$ of approximations to (ϱ, f_n), where each number $(\varrho, f_n)^{(M)}$ is explicitly computable in terms of the points of period at most M.

This suggests a natural means of approximating the density function ϱ: first choose positive integers N and $M = M(N)$, next compute the numbers $(\varrho, f_n)^{(M)}$ for $0 \leq n \leq N$, then finally combine them to define

$$\varrho_N^{(M)} := \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n \, .$$

The function $\varrho_N^{(M)}$ is a closed form approximation of the true density ϱ.

This approximation scheme works exceedingly well in practice. In the first half of the article we explain in detail the mechanics of this scheme, as well as its theoretical underpinning (thermodynamic formalism and determinants of trace class operators). The approximation scheme is most successful in providing moderate estimates (double precision, say) in reasonable time. If a lower precision (eg. four correct decimal digits) is sufficient, this can be achieved essentially instantaneously, using orbits of very low period (up to period 5, say); this is illustrated by an explicit example in §4.

If much higher precision estimates on ϱ are required then the choice of N and $M = M(N)$ becomes an issue. The choice of N depends on the *decay rate* of the coefficients (ϱ, f_n) as $n \to \infty$. For suitable orthonormal bases (including the Fourier and Legendre bases, cf. Lemma 7.1) we do have some a priori control over this decay rate: it is $O(\alpha_\varrho^n)$, for some $0 < \alpha_\varrho < 1$, by virtue of the analyticity of ϱ. This means that we can use $\varrho_N^{(M)}$ to uniformly approximate ϱ, with an error which is $O(\alpha_\varrho^N)$, provided $M = M(N)$ is chosen sufficiently large. It is desirable, therefore, to have an idea of how large M should be chosen, and this motivates the following theorems.

First we estimate $M(N)$ in the case where $\{f_n\}$ is the normalised Legendre basis.

Theorem 1. *(Normalised Legendre basis)*

Suppose that T is a topologically mixing piecewise C^ω expanding Markov map on the interval, with absolutely continuous invariant measure μ, and corresponding density function ϱ. Let $\{f_n\}_{n=0}^\infty$ denote the set of normalised Legendre polynomials.

There exists $C > 0$ and a sequence of computable numbers $(\varrho, f_n)^{(M)}$, $0 \leq n \leq N$, depending on the points of period at most $M := M(N) = \exp(CN^2)$, such that the function $\varrho_{N,Legendre}^{(M)} := \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n$ is an approximation to the density ϱ satisfying the following error bound:

There exists $0 < \alpha_\varrho < 1$ and $E > 0$ such that, for all sufficiently large N,

$$\left|\left| \varrho - \varrho_{N,Legendre}^{(M)} \right|\right|_{\infty, I} \leq E \alpha_\varrho^N .$$

Next we consider circle expanding maps. In this case the Fourier basis can be used, and an improved estimate on $M(N)$ is possible.

Theorem 2. *(Fourier basis)*

Suppose that T is a C^ω expanding map of the circle, with absolutely continuous invariant measure μ, and corresponding density function ϱ. Let $f_n(x) = 2^{-1/2}e^{\pi nix}$ for $n \in \mathbb{Z}$. There exists $C > 0$ and a sequence of computable numbers $(\varrho, f_n)^{(M)}$, $-N \le n \le N$, depending on the points of period at most $M := M(N) = \exp(CN)$, such that the function $\varrho^{(M)}_{N,Fourier} := \sum_{n=-N}^{N}(\varrho, f_n)^{(M)} f_n$ is an approximation to the density ϱ satisfying the following error bound:

There exists $0 < \alpha_\varrho < 1$ and $E > 0$ such that, for all sufficiently large N,

$$\left|\left|\varrho - \varrho^{(M)}_{N,Fourier}\right|\right|_{\infty,\mathbb{T}} \le E\alpha_\varrho^N.$$

The proof of these results occupies most of the second half of the article, from §5 onwards. The two key factors influencing the efficiency of the algorithm are the amount of expansion in the dynamics and the growth properties of the holomorphically extended basis functions.

In §5 we present refined estimates on the decay rates of Taylor coefficients for certain dynamical determinants, improving on previous estimates in [Ru1], [Fried], [JP]. This decay rate is related to the expansiveness of T: the more expanding T is, the faster the decay rate, and, ultimately, the more efficient the algorithm. In §6 these estimates are applied to particular determinants Δ_n associated to the basis functions f_n, yielding concrete bounds on the differences $|(\varrho, f_n)^{(M)} - (\varrho, f_n)|$. A sequence of conformal coordinate changes is introduced so as to control the growth of the f_n in the complex plane (the need for this is illustrated by Proposition 6.1). In §7 this leads to effective bounds on the norms of associated transfer operators. In §8 these various ingredients are combined to prove the above theorems.

§9 consists of some concluding remarks about the application of the approximation scheme in the case where T is smooth but not real analytic; the algorithm can be followed verbatim, though its efficiency is somewhat reduced.

Remarks.

(1) The Legendre polynomial expansion of ϱ suggests the possibility of computing its expansion with respect to other systems of polynomials (cf. [Sz], [BB]). In general such expansions need not even exist, however. Even when they do, the corresponding expansion coefficients need not be computable by integration along I, in which case their calculation is not amenable to our method. Such difficulties are present even for the case of the Taylor expansion of ϱ.

(2) Other authors have used estimates for the density to derive estimates on the metric entropy $h(\mu)$ via the Rohlin-Pesin equality [Pe]. However, we recall that a more efficient method is given in [JP], working directly with the periodic point data.

2. Integrals of individual functions

Suppose the interval is partitioned as $I = I_1 \cup \ldots \cup I_d$, $d \ge 2$, where $I_i = [x_{i-1}, x_i]$, and $-1 = x_0 < x_1 < \ldots < x_d = 1$. Given $T : I \to I$, we shall always assume that $T|_{I_i}$ is real analytic, for each $1 \le i \le d$. We say T is *piecewise expanding* if there exists $\Lambda > 1$ such that $|T'|\,|_{I_i} \geq \Lambda$ for all $1 \leq i \leq d$. We say that T is *Markov* if there exists a $d \times d$ matrix A (the *transition matrix* for T) with each entry either 0 or 1, such that $T(I_i) = \cup_{j:A(i,j)=1} I_j$ for each $1 \leq i \leq d$. The collection $\{I_i\}_{i=1}^d$ is called the *Markov partition* for T. T is *topologically mixing* if some power of the transition matrix A is a strictly positive matrix.

It is well known (see [LY]) that any topologically mixing piecewise C^ω expanding Markov map admits a unique ergodic absolutely continuous invariant probability measure. Our results are valid for all such maps, though to simplify the exposition we shall always assume that T is a *full branch* expanding map. In other words, each $T|_{I_i}$ is assumed to be a surjection onto I, or, equivalently, every entry of the corresponding transition matrix A is a 1.

Let μ denote the unique ergodic absolutely continuous T-invariant probability measure, and let $\varrho \in C^\omega$ (cf. [JP, Lemma 1]) denote the associated density i.e., $d\mu(x) = \varrho(x)dx$.

Any C^ω expanding (i.e. $|T'| > 1$) map of the circle is a covering map (see [Sh]). If $d \geq 2$ is the absolute value of its degree, then there is a natural identification with a full branch piecewise expanding interval map whose Markov partition has cardinality d. In particular, both maps have the same density function ϱ. If ϱ is regarded as a map defined on the interval, then $\varrho(-1) = \varrho(1)$. Later (see §3 in particular) we shall exploit this by expanding ϱ with respect to an orthonormal basis for $L^2(\mathbb{T}, dx)$ (rather than for the larger space $L^2(I, dx)$). Usually, however, we shall always assume that T is a piecewise C^ω full branch expanding map on the *interval*.

We recall that in [JP] a sequence of invariant signed measures ν_M was defined by

$$\nu_M = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m \leq M}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x \in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1\\ j\neq i}}^m \sum_{y \in \mathrm{Fix}(n_j)} r(y, n_j) \right) \frac{\delta_x}{|(T^{n_i})'(x) - 1|}$$

where the first summation is over ordered m-tuples of positive integers whose sum is not greater than M, where $\mathrm{Fix}(n)$ denotes the set of fixed points of T^n, and where

$$r(x, n) = \frac{1}{n|(T^n)'(x) - 1|}.$$

The measure ν_M is supported on those periodic points of period at most M. Introducing the normalisation constant

$$I_M = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m \leq M}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{x \in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1\\ j\neq i}}^m \sum_{y \in \mathrm{Fix}(n_j)} r(y, n_j) \right) \frac{1}{|(T^{n_i})'(x) - 1|},$$

we then define the invariant signed probability measures $\mu_M = I_M^{-1} \nu_M$, and prove [JP]:

Proposition 2.1. *Suppose that $T : I \to I$ is a topologically mixing piecewise C^ω expanding Markov map of the interval, with absolutely continuous invariant probability measure μ. There is a sequence of T-invariant signed probability measures μ_M, supported on the points* *of period at most M, such that for every C^ω function $g : I \to \mathbb{R}$, there exists* $0 < \theta < 1$ *and* $C > 0$ *with*

$$\left| \int g \, d\mu_M - \int g \, d\mu \right| \leq C\theta^{M(M+1)/2}.$$

An alternative way to view the formulae defining μ_M, ν_M is in terms of the Taylor coefficients of certain entire functions $\Delta : \mathbb{C} \to \mathbb{C}$, $\Delta_g : \mathbb{C}^2 \to \mathbb{C}$, defined by

$$\Delta(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}(n)} \frac{1}{|(T^n)'(x) - 1|} \ , \tag{2.1}$$

$$\Delta_g(z,t) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}(n)} \frac{\exp\left(tg^n(x)\right)}{|(T^n)'(x) - 1|} \tag{2.2}$$

for sufficiently small values of z, and then by analytic continuation. Here $g^n(x)$ denotes the sum $\sum_{r=0}^{n-1} g(T^r x)$. If

$$\Delta(z) = 1 + \sum_{m=1}^{\infty} c_m z^m \quad , \quad \Delta_g(z,t) = 1 + \sum_{m=1}^{\infty} c_{m,g}(t) z^m$$

are the corresponding Taylor expansions, then (see [JP], or Proposition 2.2 below) we have

$$\int g \, d\nu_M = \sum_{m=1}^{M} c'_{m,g}(0) \quad , \quad I_M = \sum_{m=1}^{M} m c_m \ ,$$

so that

$$\int g \, d\mu_M = \frac{\sum_{m=1}^{M} c'_{m,g}(0)}{\sum_{m=1}^{M} m c_m} \ .$$

The functions Δ, Δ_g are both *determinants* of trace class transfer operators acting on a suitable Hilbert space of analytic functions (see [Sim] for the general theory of determinants associated to certain Hilbert space operators). For any C^ω function $f : I \to \mathbb{R}$, the corresponding *transfer operator* $\mathcal{L}_f$ is defined (cf. [Ru2]) by

$$\mathcal{L}_f v(x) = \sum_{Ty=x} v(y) e^{f(y)}.$$

As is well known, this operator preserves various function spaces. In particular we will consider such operators as acting on the Hilbert Bergman space $A_2(U)$ corresponding to appropriate complex domains U. Recall (see for example [HKZ]) that $A_2(U)$ is the space of functions analytic on U and such that $\int_U |f|^2 \, dA < \infty$, where A is two-dimensional Lebesgue measure on U. When equipped with inner product[3] $\langle f, g \rangle = \int_U f \bar{g} \, dA$ this is

[3] We use $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ to denote the inner product in $A_2(U)$ and $L^2(I)$ respectively

a Hilbert space. The reason we work with this particular Hilbert space is its relation to $L^2(I)$: if $U_0 \supset U_1 \supset \dots$ is a sequence of complex domains, shrinking to I, and $g \in A_2(U_n)$ for all sufficiently large n, then $A(U_n)^{-1}||g||_{A_2(U_n)} \to ||g||_{L^2(I)}$. We will occasionally also work with the space $A_\infty(U)$ of those functions which are analytic on U and bounded on its closure $\overline{U}$, equipped with the supremum norm (which makes it a Banach, but not a Hilbert, space).

The action of transfer operators on Hilbert Bergman spaces has hitherto not been considered in the literature, so we prove (Lemma 5.1) that $\mathcal{L}_f : A_2(U) \to A_2(U)$ is a trace class operator, for suitable domains U. This ensures that $\mathcal{L}_f$ has a well-defined determinant $\det(I - z\mathcal{L}_f) = \exp[\mathrm{tr}(\log(I - z\mathcal{L}_f))]$, which extends to an entire function of z (cf. [Sim, Thm. 3.3]). Now $\mathrm{tr}(\log(I - z\mathcal{L}_f)) = -\sum_{n=1}^\infty \frac{z^n}{n}\mathrm{tr}(\mathcal{L}_f^n)$ for small z, and each $\mathcal{L}_f^n$ is a sum of weighted composition operators, so its trace may be expressed (cf. [May1], [May2]) as $\mathrm{tr}(\mathcal{L}_f^n) = \sum_{x\in\mathrm{Fix}(n)} \frac{e^{f^n(x)}}{1-[(T^n)'(x)]^{-1}}$, yielding the formula

$$\det(I - z\mathcal{L}_f) = \exp - \sum_{n=1}^\infty \frac{z^n}{n} \sum_{x\in\mathrm{Fix}(n)} \frac{\exp(f^n(x))}{1-[(T^n)'(x)]^{-1}}. \tag{2.3}$$

It is then clear that $\Delta(z) = \det(I - z\mathcal{L}_{-\log|T'|})$ and $\Delta_g(z,t) = \det(I - z\mathcal{L}_{tg-\log|T'|})$.

The transfer operator corresponding to the first of these choices, $f = -\log|T'|$, is the *Perron-Frobenius operator* for the map T, and will be denoted by $\mathcal{L}$.

On the other hand these are *spectral* determinants: we have (see [Sim, Thm. 3.7]) the equality $\det(I - z\mathcal{L}_f) = \prod_{i=1}^\infty (1 - z\lambda_i(\mathcal{L}_f))$, where the infinite product is absolutely convergent and $\{\lambda_i(\mathcal{L}_f)\}_{i=1}^\infty$ is an enumeration of the non-zero eigenvalues of $\mathcal{L}_f$, each repeated according to its algebraic multiplicity. ($\mathcal{L}_f$ is compact since it is trace class, so its spectrum is countable, with zero the only point of accumulation, and all non-zero eigenvalues of finite algebraic multiplicity).

The leading (i.e. largest in modulus) eigenvalue of any transfer operator $\mathcal{L}_f$ is well known to be simple, and equal to $e^{P(f)}$, where $P(f) = P(f,T) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in\mathrm{Fix}(n)} e^{f^n(x)}$ is the *pressure* of f (cf. [Ru2]). The leading eigenvalue of the Perron-Frobenius operator $\mathcal{L}$ equals 1, since it is easily checked that $P(-\log|T'|) = 0$. The corresponding eigenfunction is precisely the invariant density ϱ (cf. [CE]).

Our motivation for introducing the functions $\Delta_g(z,t)$ is the following result (see [JP]), whose proof we provide for completeness.

Proposition 2.2. *For any real analytic function $g : I \to \mathbb{R}$,*

$$\int g\, d\mu = \frac{\partial_t \Delta_g(1,0)}{\partial_z \Delta_g(1,0)} = \frac{\partial_t \Delta_g(1,0)}{\Delta'(1)} .$$

Truncating the Taylor expansion for both numerator and denominator of the righthand side yields the sequence

$$\int g\, d\mu_M = \frac{\int g\, d\nu_M}{I_M} = \frac{\sum_{m=1}^M c'_{m,g}(0)}{\sum_{m=1}^M m c_m}$$

of approximations to $\int g\, d\mu$. *The rate of convergence of these approximations is superexponential in* M, *as described in Proposition 2.1.*

Proof. The maximal eigenvalue of the operator $\mathcal{L}_{tg-\log|T'|}$ is $\exp P(tg - \log|T'|)$. Its reciprocal $z(t) := \exp\left[-P(tg - \log|T'|)\right]$ is then a zero of $\Delta_g(z,t)$, so that $\Delta_g(z(t),t) = 0$ for all t. Moreover, we have $z(0) = \exp\left[-P(-\log|T'|)\right] = 1$, since 1 is the maximal eigenvalue of the Perron-Frobenius operator $\mathcal{L}$.

The derivative formula for the pressure functional (see [Ru2, p.133], [PP, p.60]) gives

$$\int g\, d\mu = \frac{d}{dt} P(tg - \log|T'|)\bigg|_{t=0} .$$

Since $z(0) = 1$, this gives

$$\int g\, d\mu = -\frac{d}{dt}\log z(t)\bigg|_{t=0} = -\frac{z'(0)}{z(0)} = -z'(0). \tag{2.4}$$

The implicit function theorem then gives

$$z'(0) = -\frac{\partial_t \Delta_g(z(0),0)}{\partial_z \Delta_g(z(0),0)} = -\frac{\partial_t \Delta_g(1,0)}{\partial_z \Delta_g(1,0)} = -\frac{\partial_t \Delta_g(1,0)}{\Delta'(1)}, \tag{2.5}$$

the last equality following fom the observation that $\Delta_g(z,0) = \Delta(z)$ and hence $\partial_z \Delta_g(z,0) = \Delta'(z)$. The result follows. □

3. Series expansion of the density ϱ

We will regard the density function ϱ as an L^2 function on either the interval or the circle, as appropriate, and consider its expansion with respect to a suitable orthonormal basis (i.e. complete orthonormal set). A minimum requirement will be that the conjugate (see below for the definition) of every basis function is real analytic. Since we wish to perform computations involving these functions, they will in practice be chosen to be as elementary as possible (eg. polynomials or trigonometric functions), and in particular will be entire.

A natural basis for $L^2([-1,1],dx)$, consists of normalised Legendre polynomials. These are commonly used in mathematical physics, and are generated by applying the Gram-Schmidt orthogonalisation process to the monomials $\{x^n\}_{n=0}^\infty$. The n^{th} Legendre polyonomial P_n is defined to be

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \dots\right\},$$

so that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2-1)/2$, $P_3(x) = (5x^3-3x)/2$, $P_4(x) = (35x^4 - 30x^2 + 3)/8$, etc. An ortho*normal* basis is obtained by setting $f_n = (n+\frac{1}{2})^{1/2} P_n$ (cf. [WW], 15.14). We then have the expansion $\varrho = \sum_{n=0}^\infty (\varrho, f_n) f_n$, where $(\varrho, f_n) = \int_{-1}^1 \varrho(x)\overline{f_n(x)}\, dx = \int_{-1}^1 f_n(x)\varrho(x)\, dx = \int f_n\, d\mu$.

In the case where T is an expanding map of the *circle*, its density function belongs to $L^2(\mathbb{T},dx)$, which we can regard as a subspace of $L^2(I,dx)$. The natural orthonormal basis for $L^2(\mathbb{T},dx)$ is given by $f_n(x) = 2^{-1/2}e^{\pi inx}$, $n \in \mathbb{Z}$. In this case we can expand

$$\varrho = \sum_{n=-\infty}^{\infty} (\varrho, f_n) f_n \ , \qquad \text{where} \qquad (\varrho, f_n) = \overline{(f_n, \varrho)} = 2^{-1/2} \int_{-1}^{1} \varrho(x) e^{-\pi nix}\, dx = \int \overline{f_n}\, d\mu \ .$$

More generally, let us suppose that $\{f_n\}_{n=0}^{\infty}$ is an orthonormal basis for either of the Hilbert spaces $L^2(I,dx)$ or $L^2(\mathbb{T},dx)$. Since any function in $L^2(\mathbb{T},dx)$ can be considered as a function in $L^2(I,dx)$, we shall consider each basis element as a function $f_n : I \to \mathbb{C}$. By $\overline{f_n} : I \to \mathbb{C}$ we denote the complex conjugate function given by $\overline{f_n}(x) = \overline{f_n(x)}$. We shall always require that for each basis function f_n, its conjugate $\overline{f_n}$ is real analytic (i.e. has a holomorphic extension to some neighbourhood of I in $\mathbb{C}$). For example if $f_n(x) = 2^{-1/2}e^{\pi nix}$ then $\overline{f_n}(x) = 2^{-1/2}e^{-\pi nix}$, which extends to an entire function $2^{-1/2}e^{-\pi niz}$ of the complex variable z. Note that this differs from first extending f_n to all of $\mathbb{C}$ and then conjugating, as this yields the *anti*holomorphic function $2^{-1/2}e^{-\pi ni\bar{z}}$. Of course in the case where each f_n is real-valued on I (as for the Legendre polynomials) then our requirement is simply that each f_n itself be real analytic.

The coefficients (ϱ, f_n) in the expansion $\varrho = \sum_{n=0}^{\infty}(\varrho, f_n) f_n$ are given by $(\varrho, f_n) = \int_{-1}^{1} \varrho(x)\overline{f_n}(x)\, dx = \int \overline{f_n}\, d\mu$. Therefore any particular (ϱ, f_n) may be accurately computed by setting $g = \overline{f_n}$ and applying the periodic point algorithm described in §2.

The first step in such a computation is to let $\mathcal{L}_{t,n}$ denote the transfer operators $\mathcal{L}_{t\overline{f_n} - \log|T'|}$ and define the corresponding sequence of determinants

$$\Delta_n(z,t) = \det(I - z\mathcal{L}_{t,n}) = \exp - \sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{x \in \mathrm{Fix}(k)} \frac{\exp\left(t\overline{f_n}^k(x)\right)}{|(T^k)'(x) - 1|} \ . \tag{3.1}$$

Of course each Δ_n enjoys the same properties as the more general determinants Δ_g, and in particular is an entire function of z. Let us write the Taylor expansion around zero as

$$\Delta_n(z,t) = 1 + \sum_{m=1}^{\infty} {}_n c_m(t) z^m \ .$$

The following is a reformulation of Proposition 2.2, giving an exact formula for the n^{th} expansion coefficient (ϱ, f_n), as well as a computational scheme for its approximation.

Proposition 3.1. *Let T be a topologically mixing C^ω expanding Markov map on either the interval I or the circle $\mathbb{T}$, and let ϱ be the density function for the unique absolutely continuous T-invariant probability measure μ. Let $\{f_n\}$ be an orthonormal basis for either $L^2(I,dx)$ or $L^2(\mathbb{T},dx)$ as appropriate, where the conjugate $\overline{f_n}$ of each basis element is real analytic.*

Let $\varrho = \sum_{n=0}^{\infty}(\varrho, f_n) f_n$ be the expansion of the density function with respect to this basis. Then for each n,

$$(\varrho, f_n) = \frac{\partial_t \Delta_n(1,0)}{\partial_z \Delta_n(1,0)} = \frac{\partial_t \Delta_n(1,0)}{\Delta'(1)}, \tag{3.2}$$

where Δ and Δ_n are the functions defined by (2.1) and (3.1) respectively.

Truncating the Taylor expansion for both numerator and denominator of the righthand side yields the sequence

$$(\varrho, f_n)^{(M)} := \int \overline{f_n}\, d\mu_M = \frac{\int \overline{f_n}\, d\nu_M}{I_M} = \frac{\sum_{m=1}^{M} {}_n c_m'(0)}{\sum_{m=1}^{M} m c_m} \tag{3.3}$$

of approximations to $(\varrho, f_n) = \int \overline{f_n}\, d\mu$, whose rate of convergence is superexponential in M.

In practice we can of course only compute finitely many of the expansion coefficients (ϱ, f_n). It is therefore advantageous if they decay to zero rapidly as $n \to \infty$. For *suitable* bases $\{f_n\}$ the real analyticity of ϱ is enough to ensure that this decay is exponential. Both the Legendre and Fourier bases are suitable in this sense (see Lemma 7.1), though for general bases this will be an additional hypothesis. The following result, which will be useful later, follows immediately from Proposition 3.1.

Lemma 3.2. *Let T be a topologically mixing C^ω expanding Markov map on either the interval I or the circle $\mathbb{T}$, and let ϱ be the density function for the unique absolutely continuous T-invariant probability measure μ. Let $\{f_n\}$ be an orthonormal basis for either $L^2(I, dx)$ or $L^2(\mathbb{T}, dx)$ as appropriate, where the conjugate $\overline{f_n}$ of each basis element is real analytic. Suppose in addition that there exist $0 < \alpha_\varrho < 1$ and $K_\varrho > 0$ such that $|(\varrho, f_n)| \le K_\varrho \alpha_\varrho^n$ for all $n \ge 0$.*

Then $\partial_t \Delta_n(1,0) = O(\alpha_\varrho^n)$ as $n \to \infty$.

4. An example

We have described the way in which each expansion coefficient (ϱ, f_n) of the density function ϱ may be approximated by a sequence $(\varrho, f_n)^{(M)}$ of computable numbers. We now illustrate, by means of an explicit example, the way in which this leads to a uniform approximation of ϱ.

Our expanding map will be $T(x) = 2x + \frac{1}{4\pi}\sin 2\pi x \pmod 1$. The density function ϱ is not known explicitly, though it has been approximated in [Mur] (see also [JP] for related calculations). As T is an expanding map of the circle[4], it is natural to expand ϱ with respect to the Fourier basis $f_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$. We clearly have $(\varrho, f_0) = \int \varrho(x)\, dx = 1$, since ϱ is a density function.

Now $|T'(x)| = 2 + \frac{1}{2}\cos 2\pi x$ is symmetric about the point $x = 1/2$. Therefore the Perron-Frobenius operator preserves the space of functions which are symmetric about $1/2$, and its fixed point ϱ will belong to this space. Consequently its Fourier coefficients are symmetric: $(\varrho, f_n) = (\varrho, f_{-n})$ for all $n \in \mathbb{Z}$. In other words, ϱ can be expanded in terms of $\{\cos 2\pi n x\}_{n=0}^{\infty}$.

Suppose we wish to approximate ϱ by computing its Fourier coefficients (ϱ, f_n) for $-2 \le n \le 2$. By the above discussion it suffices to approximate the coefficients (ϱ, f_1) and (ϱ, f_2). These approximations will be derived from the periodic orbit data of T. Clearly T has a single fixed point, at zero. The other periodic orbits can be located numerically. For

[4] with circumference 1, in contrast to the parametrisation used in other sections

example there is a single period-2 orbit, $\{0.308584\ ,\ 0.691415\}$, and two period-3 orbits, $\{0.098318\ ,\ 0.242727\ ,\ 0.564949\}$ and $\{0.435050\ ,\ 0.901681\ ,\ 0.757272\}$.

Define

$$a_k = \frac{1}{k} \sum_{x\in \mathrm{Fix}(k)} \frac{1}{|(T^k)'(x) - 1|}\ , \tag{4.1}$$

so that $\Delta(z) = \exp - \sum_{k=1}^{\infty} a_k z^k$. We previously wrote $\Delta(z) = 1 + \sum_{m=1}^{\infty} c_m z^m$, so comparing coefficients in the two expressions yields

$$\begin{gathered} c_1 = -a_1 \quad , \quad c_2 = -a_2 + \frac{1}{2}a_1^2 \quad , \quad c_3 = -a_3 + a_1 a_2 - \frac{1}{6}a_1^3 \\ c_4 = -a_4 - \frac{1}{2}a_1^2 a_2 + a_1 a_3 + \frac{1}{2}a_2^2 + \frac{1}{24}a_1^4\ , \quad \text{etc.} \end{gathered} \tag{4.2}$$

Define

$$_n b_k = \frac{1}{k} \sum_{x\in \mathrm{Fix}(k)} \frac{\overline{f}_n{}^k(x)}{|(T^k)'(x) - 1|}\ , \tag{4.3}$$

and observe that

$$\begin{aligned} \partial_t \Delta_n(z,0) &= \left(- \sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{x\in \mathrm{Fix}(k)} \frac{\overline{f}_n{}^k(x)}{|(T^k)'(x) - 1|} \right) \exp \left(- \sum_{k=1}^{\infty} \frac{z^k}{k} \sum_{x\in \mathrm{Fix}(k)} \frac{1}{|(T^k)'(x) - 1|} \right) \\ &= \left(- \sum_{k=1}^{\infty}\ {}_n b_k z^k \right) \exp - \sum_{k=1}^{\infty} a_k z^k \\ &= \left(- \sum_{k=1}^{\infty}\ {}_n b_k z^k \right) \left(1 + \sum_{m=1}^{\infty} c_m z^m \right)\ . \end{aligned}$$

However $\partial_t \Delta_n(z,0) = \sum_{m=1}^{\infty}\ {}_n c_m'(0) z^m$ as well, so comparing coefficients we derive the useful relation

$$_n c_m'(0) = - \sum_{i=1}^{m}\ {}_n b_i\, c_{m-i}\ . \tag{4.4}$$

We are now ready to approximately compute the Fourier coefficients (ϱ, f_1) and (ϱ, f_2). Suppose we choose to use the four periodic orbits of period ≤ 3 to make these approximations. From (4.1), (4.2) we obtain the values c_1, c_2, c_3, then from (4.3), (4.4) we obtain ${}_1c_1'(0)$, ${}_1c_2'(0)$, ${}_1c_3'(0)$ and ${}_2c_1'(0)$, ${}_2c_2'(0)$, ${}_2c_3'(0)$. By Proposition 3.1 we therefore have

$$(\varrho, f_1) \approx (\varrho, f_1)^{(3)} = \frac{\sum_{m=1}^{3}\ {}_1c_m'(0)}{\sum_{m=1}^{3} m c_m} = \frac{-0.02922276}{-1.36108207} = 0.021470$$

and

$$(\varrho, f_2) \approx (\varrho, f_2)^{(3)} = \frac{\sum_{m=1}^{3}\ {}_2c_m'(0)}{\sum_{m=1}^{3} m c_m} = \frac{0.05752538}{-1.36108207} = -0.042264\ .$$

Our approximate density function is therefore

$$\varrho_2^{(3)}(x) = 1 \; + \; 0.04294 \cos 2\pi x \; - \; 0.084528 \cos 4\pi x \, .$$

A better approximation to ϱ is obtained if we use the six periodic orbits of period ≤ 4. Here we have

$$(\varrho, f_1) \approx (\varrho, f_1)^{(4)} = \frac{\sum_{m=1}^{4} \; {}_1c_m'(0)}{\sum_{m=1}^{4} mc_m} = \frac{-0.03574251}{-1.36128967} = 0.026256$$

and

$$(\varrho, f_2) \approx (\varrho, f_2)^{(4)} = \frac{\sum_{m=1}^{4} \; {}_2c_m'(0)}{\sum_{m=1}^{4} mc_m} = \frac{-0.00231526}{-1.36128967} = 0.001700 \, ,$$

so the corresponding approximate density is

$$\varrho_2^{(4)}(x) = 1 \; + \; 0.052512 \cos 2\pi x \; + \; 0.0034 \cos 4\pi x \, .$$

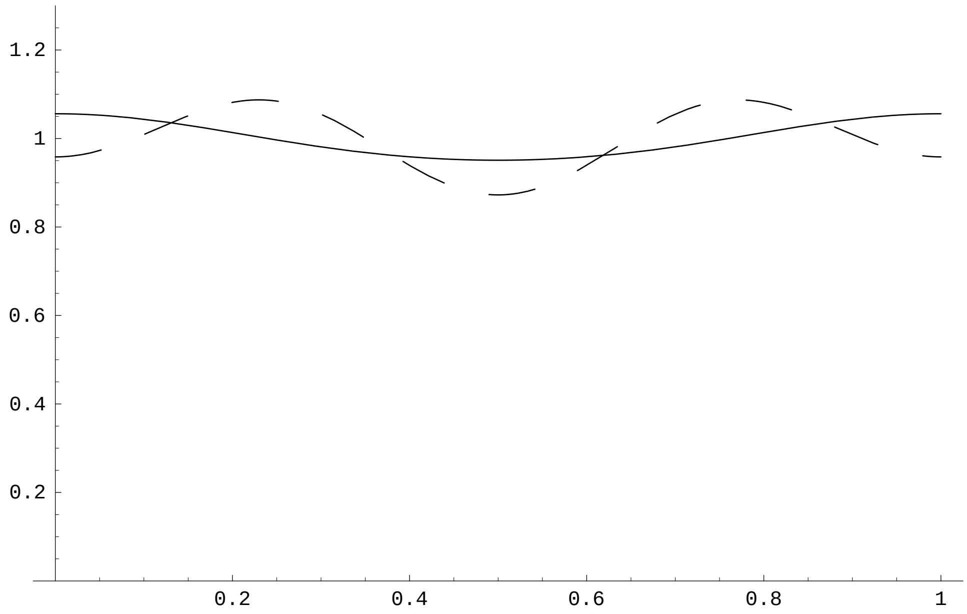

FIGURE 1. APPROXIMATE DENSITIES $\varrho_2^{(3)}$ (DASHED) AND $\varrho_2^{(4)}$

The two approximate densities $\varrho_2^{(3)}$, $\varrho_2^{(4)}$ are graphed in Figure 1. In fact $\varrho_2^{(4)}$ is already a good approximation; its graph is indistinguishable, in the resolution of Figure 1, from that of the next approximation

$$\varrho_2^{(5)}(x) = 1 \; + \; 0.052458 \; \cos 2\pi x \; + \; 0.004252 \cos 4\pi x \, .$$

The above computations require a few minutes with a pocket calculator, or a fraction of a second with a modern computer.

The fact that each $\varrho_n^{(M)}$ is in closed form means we can then rapidly evaluate $\int g\,d\mu = \int g(x)\varrho(x)\,dx$ for other integrable functions g. Of particular interest is the choice $g(x) = \log|T'(x)| = \log(2+\frac{1}{2}\cos 2\pi x)$, since $h(\mu) = \int \log|T'|\,d\mu$ is the metric entropy of the absolutely continuous invariant measure μ, by the Rohlin-Pesin identity [Pe]. Using the approximate density $\varrho_2^{(5)}$ we find that $h(\mu) \approx \int g(x)\varrho_2^{(5)}(x)\,dx = 0.683778$, of which the first five digits are correct (cf. Table 3 in [JP], where a more direct means of approximating $h(\mu)$ is employed).

5. Contraction ratios and trace class estimates

We now begin an investigation into the efficiency of our approximation scheme. The first step is to consider transfer operators $\mathcal{L}_f$ with arbitrary C^ω weight functions f. More precisely, we shall regard such $\mathcal{L}_f$ as acting on suitable Hilbert Bergman spaces, and prove results about the associated determinant $\det(I - z\mathcal{L}_f)$. In §6 these results will be interpreted for weight functions whose complex conjugates belong to our orthonormal basis for $L^2(I, dx)$.

Our piecewise C^ω map $T : I \to I$ is assumed to be a full branch piecewise expanding map. In particular, each $T|_{I_i} : I_i \to I$ is a C^ω diffeomorphism, so there is a C^ω *inverse branch* $T_i : I \to I_i$ such that $T_i \circ T$ is the identity on I, and moreover $||T_i'||_{\infty,I} < 1$ for each $1 \le i \le d$.

Definition. (Admissible domain)
The real analyticity hypothesis means we can choose a simply connected complex domain $U \supset I$ such that each inverse branch T_i has a holomorphic extension (also denoted T_i) to U, such that $||T_i'||_{\infty,U} < 1$ for each $i = 1, \dots, d$. Such a domain U will be called *admissible*. If U is an admissible domain then $\cup_{i=1}^d T_i(U)$ is compactly contained in U.

If $f : I \to \mathbb{C}$ is real analytic, and such that $\exp(f)$ has a holomorphic extension to some admissible domain U for T, then we say that U is an *admissible domain for the pair* (T, f).

We note that any sufficiently small complex neighbourhood of I is admissible in both of the above senses.

From Proposition 2.2 we know that the rate at which the approximations $\int g\,d\mu_M$ approach $\int g\,d\mu$ depends upon the speed with which the Taylor coefficients c_m and $c'_{m,g}(0)$ decay to zero. Roughly speaking, the decay rate of these coefficients can be estimated by comparing the size of an admissible domain U to its image $\cup_{i=1}^d T_i(U)$ under the inverse branches of T. This is most clear cut in the case where U can be chosen as a round (i.e. Euclidean) disc containing I. Let $\tilde{U}$ denote the smallest disc, concentric with U, which contains $\cup_{i=1}^d T_i(U)$. If $\theta < 1$ denotes the ratio of the radii of $\tilde{U}$ and U, then this turns out to be the same θ as in Proposition 2.1.

In general, however, it is not possible to choose U to be a round disc: there may be singularities (of either e^f or of an inverse branch T_i) close to I in the complex plane. Nevertheless, for technical reasons (see the proof of Lemma 8 in [JP], or of Lemma 5.1 below) it is important to work with round discs. In [JP] this problem was circumvented by working with a *sufficiently refined* Markov partition, then choosing a small complex disc around each partition piece. In this article we shall use a different approach, working with general (simply connected) domains U but then mapping them biholomorphically onto the open unit disc.

Definition (Contraction ratio). We say that $\tau \in (0,1)$ is a *contraction ratio* associated to the admissible domain U for the map T if

$$\cup_{i=1}^d T_i(U) \subset \tau U \ ,$$

where $\tau U = \{\tau u : u \in U\}$.

Now suppose $U_0 \supset U_1 \supset \ldots$ is a sequence of admissible domains, with vertical height decreasing to zero, and horizontal semi-width strictly decreasing to some $W > 1$. We say that $\tau \in (0,1)$ is a *contraction ratio* for this sequence of domains if it is a contraction ratio for each U_n. The fact that each T_i is a strict contraction ensures that each union $\cup_{i=1}^d T_i(U_n)$ is *compactly contained* in U_n. Therefore every such sequence of domains has a contraction ratio τ which is strictly smaller than 1.

Definition (Disc contraction ratio). We say that $\theta \in (0,1)$ is a *disc contraction ratio* associated to the admissible domain U for the map T if there is a holomorphic diffeomorphism $\varphi : \mathbb{D} \to U$ sending the open unit disc onto U such that $\varphi^{-1}(\cup_{i=1}^d T_i(U))$ is contained in the open disc $\mathbb{D}_\theta$ of radius θ centred at 0.

The Riemann mapping φ is only unique up to pre-composition with a disc automorphism. The smallest possible disc contraction ratio for the domain U is obtained by choosing $\theta = \theta_\varphi$ as small as possible for each Riemann mapping φ, and then minimising over all φ. The smallest possible disc contraction ratio for the map T is then obtained by minimising this quantity over all admissible domains U. Such optimality in the choice of disc contraction ratio can be useful in concrete estimates for particular maps T, though in the sequel we shall not aim for such sharpness.

Given a compact operator $L : H \to H$ on a Hilbert space H, its i^{th} *approximation number* (or *singular value*) $s_i(L)$ is defined as

$$s_i(L) = \inf\{||L - K|| : \text{rank}(K) \leq i - 1\}. \tag{5.1}$$

For $p > 0$, define the *trace ideal* $\mathcal{S}_p(H)$ to consist of those compact operators $L : H \to H$ for which $\sum_i s_i(L)^p < \infty$. $\mathcal{S}_1(H)$ is known as the *trace class ideal*, and operators in $\mathcal{S}_1(H)$ are called *trace class*. Any trace class operator L has a *determinant*, defined for small z by $\exp[\text{tr}(\log(I - zL))]$, and admitting an analytic continuation as an entire function of z. Its zeros are the reciprocals of the non-zero eigenvalues of L, the order of any zero being the algebraic multiplicity of the corresponding eigenvalue.

The next lemma gives useful bounds on approximation numbers for transfer operators acting on Hilbert Bergman spaces, and in particular implies that such operators belong to every trace ideal $\mathcal{S}_p(A_2)$, $p > 0$.

For economy of notation we shall write $||\cdot||_{\theta,1}$ to denote $||\cdot||_{A_2(\mathbb{D}_\theta),A_2(\mathbb{D})}$, the operator norm of operators $A_2(\mathbb{D}_\theta) \to A_2(\mathbb{D})$, and $||\cdot||_{\theta,\theta}$ to denote $||\cdot||_{A_2(\mathbb{D}_\theta),A_2(\mathbb{D}_\theta)}$, the operator norm of operators $A_2(\mathbb{D}_\theta) \to A_2(\mathbb{D}_\theta)$, We also use $(\cdot,\cdot)_{A_2(\mathbb{D})}$ to denote the inner product in $A_2(\mathbb{D})$.

Lemma 5.1. *Let $T : I \to I$ be a topologically mixing piecewise C^ω expanding Markov map on the interval I. Let $U \subset \mathbb{C}$ be an admissible domain for the pair (T, f).*

The transfer operator $\mathcal{L}_f$ preserves the space $A_2(U)$, and $\mathcal{L}_f : A_2(U) \to A_2(U)$ is a trace class operator. In fact if $0 < \theta < 1$ is a disc contraction ratio for (T, U), then the j^{th} singular value $s_j(\mathcal{L}_f)$ satisfies

$$s_j(\mathcal{L}_f) \leq \frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\theta^j,$$

for all $j \in \mathbb{N}$.

Proof. We may suppose that conjugation by the Riemann mapping $\varphi : \mathbb{D} \to U$ has already been performed. That is, we suppose that $S = \varphi^{-1} \circ T \circ \varphi$ is a full branch expanding map on $\varphi^{-1}(I)$, with local inverse branches $S_1, \dots, S_d$, such that $\cup_{i=1}^d S_i(\mathbb{D}) \subset \mathbb{D}_\theta$, the disc of radius θ centred at 0. We therefore wish to estimate the singular values of $\mathcal{L}_f : A_2(\mathbb{D}_\theta) \to A_2(\mathbb{D}_\theta)$.

Suppose $v \in A_2(\mathbb{D}_\theta)$. Since v is holomorphic on $\mathbb{D}_\theta$ then it is also holomorphic on $\cup_{i=1}^d S_i(\mathbb{D})$. It follows that $\mathcal{L}_f v$ is holomorphic on an open neighbourhood of the unit disc $\mathbb{D}$, and in particular belongs to $A_2(\mathbb{D})$. We may write

$$\mathcal{L}_f v = \sum_{k=0}^\infty l_k(v) p_k,$$

where $p_k(z) = z^k$, and $l_k(v) = \frac{1}{k!}(\mathcal{L}_f v)^{(k)}(0)$.

An easy calculation gives

$$||p_k||_{A_2(\mathbb{D}_\theta)} = \sqrt{\frac{\pi}{k+1}}\theta^{k+1}, \tag{5.2}$$

and similarly

$$||p_k||_{A_2(\mathbb{D})} = \sqrt{\frac{\pi}{k+1}}\,. \tag{5.3}$$

We now seek to estimate $|l_k(v)|$. The monomials $\{p_j\}$ are an orthogonal family in $A_2(\mathbb{D})$, so we have $\langle \mathcal{L}_f v, p_k \rangle_{A_2(\mathbb{D})} = l_k(v)||p_k||^2_{A_2(\mathbb{D})}$. But $|\langle \mathcal{L}_f v, p_k \rangle_{A_2(\mathbb{D})}| \leq ||\mathcal{L}_f v||_{A_2(\mathbb{D})} ||p_k||_{A_2(\mathbb{D})}$, by the Cauchy-Schwarz inequality, so from (5.3) we have

$$|l_k(v)| \leq ||\mathcal{L}_f v||_{A_2(\mathbb{D})}\, ||p_k||^{-1}_{A_2(\mathbb{D})} = ||\mathcal{L}_f v||_{A_2(\mathbb{D})} \sqrt{\frac{k+1}{\pi}}. \tag{5.4}$$

Let us define the rank-j operator $\mathcal{L}_f^{(j)}$ by $\mathcal{L}_f^{(j)} = \sum_{k=0}^{j-1} l_k(v) p_k$. For any $v \in A_2(\mathbb{D}_\theta)$ we can use (5.2) and (5.4) to estimate

$$\begin{aligned}
|| \left(\mathcal{L}_f - \mathcal{L}_f^{(j)}\right)(v)||_{A_2(\mathbb{D}_\theta)} &\leq \sum_{k=j}^\infty |l_k(v)|\, ||p_k||_{A_2(\mathbb{D}_\theta)} \\
&\leq ||\mathcal{L}_f v||_{A_2(\mathbb{D})} \sum_{k=j}^\infty \theta^{k+1} \\
&\leq ||\mathcal{L}_f||_{\theta,1} ||v||_{A_2(\mathbb{D}_\theta)} \frac{\theta^{j+1}}{1-\theta}.
\end{aligned}$$

It follows that

$$||\mathcal{L}_f - \mathcal{L}_f^{(j)}||_{\theta,\theta} \le \frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\theta^{j+1}.$$

Since $\mathcal{L}_f^{(j)}$ is of rank j then we can estimate the $(j+1)^{st}$ approximation number by

$$s_{j+1}(\mathcal{L}_f) \le \frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\theta^{j+1},$$

and the result follows. □

The importance of the exponential decay of the singular values of $\mathcal{L}_f$ is that we can deduce the *superexponential* decay of the Taylor coefficients $c_m(f)$ of the corresponding determinant $\det(I - z\mathcal{L}_f)$, as described in the following lemma. This estimate is a sharpening of the inequality (3.7) in [JP].

Lemma 5.2. *Let $T : I \to I$ be a topologically mixing piecewise C^ω expanding Markov map on the interval I, and suppose $f : I \to \mathbb{C}$ is real analytic. Let $U \subset \mathbb{C}$ be an admissible domain for the pair (T, f).*

The determinant $\det(I - z\mathcal{L}_f)$ of the trace class operator $\mathcal{L}_f : A_2(U) \to A_2(U)$ is an entire function of z. Indeed if we write $\det(I - z\mathcal{L}_f) = 1 + \sum_{m=1}^\infty c_m(f) z^m$, then

$$|c_m(f)| \le B(\theta)\left(\frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\right)^m \theta^{m(m+1)/2} \tag{5.5}$$

where $0 < \theta < 1$ is a disc contraction ratio for (T, U), and $B(\theta) = \prod_{m=1}^\infty (1-\theta^m)^{-1} < \infty$.

Proof. By [Sim, Lemma 3.3] we have $c_m(f) = \sum_{i_1 < \ldots < i_m} \lambda_{i_1}(\mathcal{L}_f) \cdots \lambda_{i_m}(\mathcal{L}_f)$, the summation being over all m-tuples $(i_1, \ldots, i_m)$ of positive integers satisfying $i_1 < \ldots < i_m$. Therefore

$$\begin{aligned}
|c_m(f)| &= \left| \sum_{i_1 < \ldots < i_m} \lambda_{i_1}(\mathcal{L}_f) \cdots \lambda_{i_m}(\mathcal{L}_f) \right| \\
&\le \sum_{i_1 < \ldots < i_m} s_{i_1}(\mathcal{L}_f) \cdots s_{i_m}(\mathcal{L}_f) \\
&\le \left(\frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\right)^m \sum_{i_1 < \ldots < i_m} \theta^{i_1 + \ldots + i_m} \\
&= \left(\frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\right)^m \frac{\theta^{m(m+1)/2}}{(1-\theta)(1-\theta^2)\cdots(1-\theta^m)}. \\
&\le B(\theta)\left(\frac{||\mathcal{L}_f||_{\theta,1}}{1-\theta}\right)^m \theta^{m(m+1)/2}
\end{aligned}$$

The first inequality is the content of Corollary VI.2.6 of [GGK]. The second inequality is by Lemma 5.1, while the penultimate step is an m-fold geometric summation. □

Remark. As already mentioned, Lemma 5.2 sharpens the inequality (3.7) of [JP], which was itself based on [Ru1, p. 236] (cf. [Fr, p. 505–507], [Ru3, Remark 6.7] for corrections to the estimate in higher dimensions). The most significant improvement is the absence of any $m^{m/2}$ term. The reason for this is that here the operator $\mathcal{L}_f$ is acting on a *Hilbert*, rather than simply a Banach, space.

There are other ways in which our method here differs from that of [JP]. For example there we used a sufficiently refined Markov partition and a complexification consisting of small round discs, whereas here we use the original Markov partition and a complexification to *any* admissible domain, followed by a Riemann mapping to the unit disc. In practice this usually leads to an improved estimate on the disc contraction ratio θ.

6. Conformal coordinate changes

Suppose a piecewise C^ω expanding Markov map $T : I \to I$ is given. As mentioned previously, in general it is not possible to choose a Euclidean disc as an admissible domain. In fact from a technical viewpoint it may be preferable to choose the admissible domain U to be a rather *thin* region. This will often give superior estimates on the operator norm $||\mathcal{L}_f||_{A_2(U)}$, which is roughly proportional to the uniform norm $||f||_{A_\infty(U)}$ (cf. Lemmas 7.2 and 7.3). If f grows rapidly in the imaginary direction, say, then choosing a thin "cigar-shaped" domain U will reduce $||\mathcal{L}_f||_{A_2(U)}$, and may give improved bounds on the righthand side of (5.5).

In fact in this article we are not just considering a single function f, but rather a *sequence* $\{f_n\}$ of analytic functions forming an L^2-orthonormal basis. To simultaneously control the uniform norms of these functions, and hence the operator norms of $\mathcal{L}_{t,n}$, it is helpful to use a *decreasing sequence* U_n of admissible domains. The rationale for this is given by the following observation.

Proposition 6.1. *Let $\{f_n\}$ be an orthonormal basis for $L^2(I, dx)$. Suppose there is some open complex neighbourhood U of the interval I such that each f_n extends holomorphically to U. Then the sequence $||f_n||_{A_\infty(U)}$ is unbounded.*

Proof. Let us suppose that the sequence $||f_n||_{A_\infty(U)}$ is bounded.

Choose $K \subset U$ to be compact, with non-empty interior, and containing the interval I. By Montel's theorem, the sequence $f_n|_K$ has a subsequence $f_{n_i}|_K$ which converges uniformly on K to a function f which is holomorphic on the interior of K. In particular f is real analytic on I, and so $f \in L^2(I, dx)$.

Suppose the integer $k \geq 0$ is given. Then for all $n_i > k$ we have

$$\begin{aligned}
|(f, f_k)|^2 &= |(f, f_k) - (f_{n_i}, f_k)|^2 \\
&= |(f - f_{n_i}, f_k)|^2 \\
&\leq ||f - f_{n_i}||_{L^2(I)} ||f_k||_{L^2(I)} \\
&= ||f - f_{n_i}||_{L^2(I)} \\
&\leq \text{length}(I) ||f - f_{n_i}||_{\infty, I} \\
&\leq \text{length}(I) ||f - f_{n_i}||_{A_\infty(K)} \to 0 \text{ as } i \to \infty \, .
\end{aligned}$$

But $|(f, f_k)|$ is independent of n_i, therefore it must be identically zero. So $(f, f_k) = 0$ for all $k \geq 0$. But $\{f_k\}$ is a complete orthonormal set, hence $f|_I \equiv 0$. So $f_{n_i} \to 0$ in

$L^2(I, dx)$ as $i \to \infty$. But this is impossible because $||f_{n_i}||_{L^2(I)} = 1$ for all i, giving the required contradiction. □

The penalty we pay for working with thin cigar-shaped domains U is that the corresponding disc contraction ratio approaches 1 at a rate which is exponential in the height of U. The fact that we necessarily pay this penalty follows from estimates for the Grötzsch modulus in conformal geometry (cf. [LV, §II.2], [Dub]). In the numerical conformal mapping literature this phenomenon is known as *crowding* (cf. [DeL], [DT, §2.6], [How, Ch. 2]). For our purposes it is helpful to specify a *particular* family of biholomorphisms $\varphi_{\varepsilon,L} : \mathbb{D} \to U_{\varepsilon,L}$, whose vertical height shrinks to zero as $\varepsilon \searrow 0$, and whose horizontal axis is $[-L, L]$ for some fixed $L > 1$ chosen small enough to ensure that each $U_{\varepsilon,L}$ is an admissible domain.

Our family $\varphi_{\varepsilon,L}$ will be related to the family $\psi_r : \mathbb{D} \to \mathbb{C}$, $0 < r < 1$, defined by $\psi_r(z) = \tanh^{-1}(rz)$. The images $\psi_r(\mathbb{D})$ resemble flattened ellipses, symmetric with respect to both real and imaginary axes. Each $\psi_r(\mathbb{D})$ is contained in the vertical strip $-1 < \mathrm{Im}(w) < 1$, and its horizontal width tends to infinity as $r \nearrow 1$.

If $\gamma_a(t) = ae^{it}$, $t \in [0, 1)$, then

$$\psi_r(\gamma_a(t)) = \frac{1}{2}\tanh^{-1}\left(\frac{2ar\cos t}{1 + a^2r^2}\right) + \frac{i}{2}\tan^{-1}\left(\frac{2ar\sin t}{1 - a^2r^2}\right).$$

In particular, setting $a = 1$, we see that the semi-height of $\psi_r(\mathbb{D})$ approaches 1 as $r \nearrow 1$, while the semi-width is asymptotic to $|\log(1 - r)|/2$.

Now define $\Phi_{\varepsilon,L}(z) = \tanh^{-1}(rz) = \psi_r(z)$, where $r = r(\varepsilon, L)$ is chosen so that

$$\frac{L}{\varepsilon} = \frac{\text{semi-width}(\psi_r(\mathbb{D}))}{\text{semi-height}(\psi_r(\mathbb{D}))}.$$

Consequently $\frac{L}{\varepsilon} \sim \frac{1}{2}|\log(1 - r)|$ as $r \nearrow 1$. That is,

$$r(\varepsilon, L) \sim 1 - e^{-2L/\varepsilon} \quad \text{as } \varepsilon \searrow 0.$$

Now define $\varphi_{\varepsilon,L}(z) = (\text{semi-width}(\psi_{r(\varepsilon,L)}(\mathbb{D})))^{-1}\Phi_{\varepsilon,L}(z)$, a homothetic scaling of $\Phi_{\varepsilon,L}(z)$ so that $\varphi_{\varepsilon,L}(\mathbb{D})$ is a thin cigar-shaped region contained in $[-L, L]\times[-\varepsilon, \varepsilon]$, with semi-width L and semi-height ε. Define $U_{\varepsilon,L} := \varphi_{\varepsilon,L}(\mathbb{D})$ for $0 < \varepsilon < \varepsilon_0$, where ε_0 is chosen small enough to ensure that each $U_{\varepsilon,L}$ in the family is admissible. We then have the following estimate on the disc contraction ratios of the domains $U_{\varepsilon,L}$ as $\varepsilon \searrow 0$.

Lemma 6.2. *(Distortion Estimate) Let $T : I \to I$ be a topologically mixing piecewise C^ω expanding Markov map on the interval I. Let $U_{\varepsilon,L} = \varphi_{\varepsilon,L}\mathbb{D} \supset I$ be the family of admissible complex domains defined above. Suppose $0 < \tau < 1$ is a contraction ratio for each $U_{\varepsilon,L}$, $0 < \varepsilon < \varepsilon_0$. Then there is a disc contraction ratio $\theta_\varepsilon \in (0, 1)$ associated to the domain $U_{\varepsilon,L}$ (for the map T), with the asymptotic*

$$\theta_\varepsilon \sim 1 - 2^{1-\tau}e^{-2\tau L/\varepsilon} \quad \textit{as } \varepsilon \searrow 0$$

Proof. Let θ_ε be the smallest disc contraction ratio associated to the Riemann mapping $\varphi_{\varepsilon,L} : \mathbb{D} \to U_{\varepsilon,L}$. If $r = r(\varepsilon, L)$ is as in the definition of $\Phi_{\varepsilon,L}$ then

$$\tau = \frac{\varphi_{\varepsilon,L}(\theta_\varepsilon)}{\varphi_{\varepsilon,L}(1)} = \frac{\Phi_{\varepsilon,L}(\theta_\varepsilon)}{\Phi_{\varepsilon,L}(1)} = \frac{\tanh^{-1}(r\theta_\varepsilon)}{\tanh^{-1}(r)} = \frac{\log\left(\frac{1+r\theta_\varepsilon}{1-r\theta_\varepsilon}\right)}{\log\left(\frac{1+r}{1-r}\right)} .$$

It follows that

$$\theta_\varepsilon = \frac{\left(\frac{1+r}{1-r}\right)^\tau - 1}{r\left[1+\left(\frac{1+r}{1-r}\right)^\tau\right]} = \frac{1}{r} - \frac{2}{r\left[1+\left(\frac{1+r}{1-r}\right)^\tau\right]} .$$

However $r = r(\varepsilon, L) \sim 1 - e^{-2L/\varepsilon}$ as $\varepsilon \searrow 0$, so that

$$\theta_\varepsilon \sim \frac{1}{1-e^{-2L/\varepsilon}} - \frac{2}{(1-e^{-2L/\varepsilon})\left[1+(2e^{2L/\varepsilon})^\tau\right]} \sim 1 - 2^{1-\tau}e^{-2\tau L/\varepsilon} ,$$

as required. $\square$

For the remainder of this section we shall fix the following:

- An orthonormal basis $\{f_n\}$ for $L^2(I)$, with the property that every $\overline{f_n} : I \to \mathbb{C}$ is real analytic, and such that there exist $0 < \alpha_\varrho < 1$ and $K_\varrho > 0$ with $|(\varrho, f_n)| \le K_\varrho \alpha_\varrho^n$ for all $n \ge 0$.
- A decreasing sequence $U_0 \supset U_1 \supset \dots$ of complex domains, where U_n is an admissible domain for the pair $(T, \overline{f_n})$.
- An increasing sequence $0 < \theta_0 < \theta_1 < \dots$, where each $0 < \theta_n < 1$ is a disc contraction ratio for the pair (T, U_n).
- $B(\theta_n) = \prod_{m=0}^\infty (1-\theta_n^m)^{-1}$, as introduced in Lemma 5.2.

Lemma 6.3. *Let $T : I \to I$ be a topologically mixing piecewise C^ω expanding Markov map. Let $\{f_n\}_{n=0}^\infty$ be an orthonormal basis for $L^2(I, dx)$, where each conjugate function $\overline{f_n}$ is real analytic. Let $U_0 \supset U_1 \supset \dots$ be a sequence of admissible domains for the pairs (T, f_n), and let θ_n be a corresponding sequence of disc contraction ratios. Set $B(\theta_n) = \prod_{m=1}^\infty (1-\theta_n^m)^{-1} < \infty$. Then*

(i) The determinant $\Delta(z) = 1 + \sum_{m=1}^\infty c_m z^m$ defined in (2.1) is an entire function of z, whose Taylor coefficients satisfy

$$|c_m| \le B(\theta_0) \left(\frac{||\mathcal{L}||_{\theta_0,1}}{1-\theta_0}\right)^m \theta_0^{m(m+1)/2} \quad \textit{for all } m \ge 0.$$

(ii) The determinants $\Delta_n(z,t) = 1 + \sum_{m=1}^\infty {}_n c_m(t) z^m$ defined in (3.1) are each entire functions of z, whose Taylor coefficients satisfy

$$|{}_n c_m(t)| \le B(\theta_n) \left(\frac{||\mathcal{L}_{t,n}||_{\theta_n,1}}{1-\theta_n}\right)^m \theta_n^{m(m+1)/2} \quad \textit{for all } m \ge 0.$$

(iii) Each $\Delta_n(z,t)$ is an entire function of t.

Proof. Parts (i) and (ii) are immediate from Lemma 5.2. Part (iii) is by [Gr, Prop. 1]. □

We can now show that both numerator and denominator in (3.3) approach the respective numerator and denominator in (3.2) super-exponentially fast in M. An important difference is that while the super-exponential decay rate of the *numerator* does depend on n, the super-exponential decay rate of the *denominator* does not.

Lemma 6.4. *Let T be a topologically mixing piecewise C^ω expanding Markov map on either the interval I or the circle $\mathbb{T}$, and let ϱ be the density function for the unique absolutely continuous T-invariant probability measure μ. Let $\{f_n\}$ be an orthonormal basis for either $L^2(I, dx)$ or $L^2(\mathbb{T}, dx)$ as appropriate, where the conjugate $\overline{f_n}$ of each basis element is real analytic.*

Let $\{U_n\}$ be a decreasing sequence of complex domains, where each U_n is an admissible domain for the pair $(T, \overline{f_n})$. Let $\{\theta_n\}$ be an increasing sequence such that each $0 < \theta_n < 1$ is a disc contraction ratio for the pair (T, U_n).

(i)

$$\left|\Delta'(1) - \sum_{m=1}^{M} mc_m\right| = \left|\sum_{m=M+1}^{\infty} mc_m\right| \le B(\theta_0) \sum_{m=M+1}^{\infty} m \left(\frac{||\mathcal{L}||_{\theta_0,1}}{1-\theta_0}\right)^m \theta_0^{m(m+1)/2}. \quad (6.1)$$

(ii) If we define $L(n,\xi) := \max_{|t|=\xi} ||\mathcal{L}_{t,n}||_{\theta_n,1}$ for any $\xi > 0$, then

$$|\partial_t \Delta_n(1,0) - \sum_{m=0}^{M-1} {}_n c'_m(0)| = |\sum_{m=M}^{\infty} {}_n c'_m(0)| \le \sum_{m=M}^{\infty} \frac{B(\theta_n)}{\xi} \left(\frac{L(n,\xi)}{1-\theta_n}\right)^m \theta_n^{m(m+1)/2}.$$

Proof.
(i) We can bound

$$\begin{aligned}\left|\Delta'(1) - \sum_{m=1}^{M} mc_m\right| &= \left|\sum_{m=M+1}^{\infty} mc_m\right| \\ &\le \sum_{m=M+1}^{\infty} m|c_m| \le \sum_{m=M+1}^{\infty} mB(\theta_0) \left(\frac{||\mathcal{L}||_{\theta_0,1}}{1-\theta_0}\right)^m \theta_0^{m(m+1)/2}\end{aligned}$$

using the bound $|c_m| \le B(\theta_0) \left(\frac{||\mathcal{L}||_{\theta_0,1}}{1-\theta_0}\right)^m \theta_0^{m(m+1)/2}$ in Lemma 6.3 (i).

(ii) Let Γ_ξ be a circle of radius $\xi > 0$ centred at 0. Cauchy's theorem gives

$$\begin{aligned}|{}_n c'_m(0)| &= \left|\frac{1}{2\pi i}\int_{\Gamma_\xi} \frac{{}_n c_m(t)}{t^2} dt\right| \\ &\le \frac{2\pi\xi}{2\pi}\xi^{-2} \max_{t\in\Gamma_\xi} |{}_n c_m(t)| \\ &\le \frac{B(\theta_n)}{\xi}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^m \theta_n^{m(m+1)/2}\end{aligned}$$

using Lemma 6.3 (ii). This leads to the bound

$$\begin{aligned}|\partial_t\Delta_n(1,0) - \sum_{m=1}^{M-1} {}_nc'_m(0)| &= |\sum_{m=M}^{\infty} {}_nc'_m(0)| \\ &\leq \sum_{m=M}^{\infty} \frac{B(\theta_n)}{\xi}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^m \theta_n^{m(m+1)/2}. \quad \square\end{aligned}$$

We now arrive at a very explicit estimate on the difference between the expansion coefficients (ϱ, f_n) and their approximants $(\varrho, f_n)^{(M)}$.

Lemma 6.5. *Let T be a topologically mixing piecewise C^ω expanding Markov map on either the interval I or the circle $\mathbb{T}$, and let ϱ be the density function for the unique absolutely continuous T-invariant probability measure μ. Let $\{f_n\}$ be an orthonormal basis for either $L^2(I,dx)$ or $L^2(\mathbb{T},dx)$ as appropriate, where the conjugate $\overline{f_n}$ of each basis element is real analytic. Suppose there exist $0<\alpha_\varrho<1$ and $K_\varrho>0$ such that $|(\varrho,f_n)|\leq K_\varrho\alpha_\varrho^n$ for all $n\geq 0$.*

Let $\{U_n\}$ be a decreasing sequence of complex domains, with each U_n an admissible domain for the pair $(T,\overline{f_n})$. Let $\theta_0<\theta_1<\dots$ be an increasing sequence such that each $0<\theta_n<1$ is a disc contraction ratio for the pair (T,U_n).

For all $\xi>0$, all $n\geq 0$, and all sufficiently large $M\in\mathbb{N}$,

$$\left|(\varrho,f_n)^{(M)}-(\varrho,f_n)\right| \leq \frac{1}{\kappa}\left(\frac{B(\theta_n)}{\xi}\left[\frac{\theta_n^{M(M+1)/2}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^M}{1-\left(\frac{L(n,\xi)}{1-\theta_n}\right)\theta_n^{M/2}}\right]+\lambda(M)K_\varrho\,\alpha_\varrho^n\right),$$

where $\kappa>0$ and $\lambda(M)=O(\beta^{M^2})$ as $M\to\infty$, for some $0<\beta<1$.

Proof. Using Proposition 3.1 we can estimate

$$\begin{aligned}\left|(\varrho,f_n)^{(M)}-(\varrho,f_n)\right| &= \left|\frac{\int\overline{f_n}\,d\nu_M}{I_M}-\int\overline{f_n}\,d\mu\right| \\ &= \left|\frac{\sum_{m=1}^M {}_nc'_m(0)}{\sum_{m=1}^M mc_m}-\frac{\partial_t\Delta_n(1,0)}{\Delta'(1)}\right| \\ &= \left|\frac{\sum_{m=1}^M {}_nc'_m(0)}{\sum_{m=1}^M mc_m}-\frac{\sum_{m=1}^\infty {}_nc'_m(0)}{\sum_{m=1}^\infty mc_m}\right| \\ &\leq \frac{|\Delta'(1)||\sum_{m=M+1}^\infty {}_nc'_m(0)|+|\partial_t\Delta_n(1,0)||\sum_{m=M+1}^\infty mc_m|}{|\Delta'(1)|\,|\sum_{m=1}^M mc_m|} \\ &\leq \frac{\sum_{m=M+1}^\infty |{}_nc'_m(0)|}{|\sum_{m=1}^M mc_m|}+|(\varrho,f_n)|\frac{\sum_{m=M+1}^\infty |mc_m|}{|\sum_{m=1}^M mc_m|} \\ &\leq \frac{1}{|\sum_{m=1}^M mc_m|}\left(\sum_{m=M+1}^\infty |{}_nc'_m(0)|+K_\varrho\alpha_\varrho^n\sum_{m=M+1}^\infty m|c_m|\right).\end{aligned} \tag{6.2}$$

By Lemma 6.4 (i) we know that $\sum_{m=1}^{M} mc_m \to \Delta'(1) \neq 0$ as $M \to \infty$, so we can choose $M_0 \in \mathbb{N}$ such that $\kappa := \inf_{M \geq M_0} \sum_{m=1}^{M} mc_m$ is strictly positive.

If $M \geq M_0$ then (6.2) gives

$$
\begin{aligned}
&\left|(\varrho, f_n)^{(M)} - (\varrho, f_n)\right| \\
&\quad \leq \frac{1}{\kappa}\left(\sum_{m=M+1}^{\infty} |{}_n c'_m(0)| + K_\varrho \alpha_\varrho^n B(\theta_0) \sum_{m=M+1}^{\infty} m \left(\frac{||\mathcal{L}||_{\theta_0,1}}{1-\theta_0}\right)^m \theta_0^{m(m+1)/2}\right) \\
&\quad \leq \frac{1}{\kappa}\left(\left[\sum_{m=M+1}^{\infty} \frac{B(\theta_n)}{\xi}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^m \theta_n^{m(m+1)/2}\right] + \lambda(M) K_\varrho \alpha_\varrho^n\right) \\
&\quad \leq \frac{1}{\kappa}\left(\frac{B(\theta_n)}{\xi}\left[\frac{\theta_n^{M(M+1)/2}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^M}{1-\left(\frac{L(n,\xi)}{1-\theta_n}\right)\theta_n^{M/2}}\right] + \lambda(M) K_\varrho \alpha_\varrho^n\right),
\end{aligned}
$$

as required. The first inequality in this final calculation is by Lemma 6.4 (i). For the second inequality we use Lemma 6.4 (ii), and set $\lambda(M) := B(\theta_0) \sum_{m=M+1}^{\infty} m \left(\frac{||\mathcal{L}||_{\theta_0,1}}{1-\theta_0}\right)^m \theta_0^{m(m+1)/2}$. It is clear that $\lambda(M)$ is independent of n, and has the asymptotic $\lambda(M) = O(\beta^{M^2})$ as $M \to \infty$ for some $\beta < \theta_0^{1/2}$. The final inequality is a crude bound using summation of a (larger) geometric series. $\square$

7. Transfer operator norms

Our estimates in §6 are valid for a wide class of orthonormal bases $\{f_n\}$ for $L^2(I, dx)$ or $L^2(\mathbb{T}, dx)$. In this section we specialise to the case where the basis for $L^2(I, dx)$ consists of normalised Legendre polynomials, and the basis for $L^2(\mathbb{T}, dx)$ consists of Fourier characters. The normalised Legendre polynomials $f_n = \sqrt{n + \frac{1}{2}} P_n$ are naturally indexed by the non-negative integers. If we write $e_n(x) = 2^{-1/2} e^{\pi n i x}$ then these Fourier characters are indexed by the integers. In order to harmonise our notation, we let $\{f_n\}_{n=0}^{\infty}$ be the re-ordering of the e_n defined by $f_{2n} = e_{-n}$ and $f_{2n+1} = e_n$.

Our first observation is that for either basis, the corresponding expansion coefficients (ϱ, f_n) of the density function ϱ decay to zero at an exponential rate.

Lemma 7.1. *Let $\{f_n\}_{n=0}^{\infty}$ denote either the orthonormal basis for $L^2(I, dx)$ consisting of normalised Legendre polynomials, or the orthonormal basis for $L^2(\mathbb{T}, dx)$ consisting of Fourier characters, re-ordered as above. For any real analytic function $\varrho : I \to \mathbb{C}$, there exist $0 < \alpha_\varrho < 1$ and $K_\varrho > 0$ such that*

$$|(\varrho, f_n)| \leq K_\varrho \, \alpha_\varrho^n \quad \textit{for all } n \geq 0,$$

and hence there exists $\tilde{K}_\varrho > 0$ such that

$$\left|\left|\varrho - \sum_{n=0}^{N} (\varrho, f_n) f_n\right|\right|_{\infty, I} \leq \tilde{K}_\varrho \, \alpha_\varrho^N \quad \textit{for all } N \geq 0.$$

Proof. The Legendre case follows from [Sz, Thm. 9.1.1], since Legendre polynomials are a special case of Jacobi polynomials (cf. [Sz, Ch. IV]). In this case the optimal choice of α_ϱ (namely $\limsup_{n\to\infty} |(\varrho, f_n)|^{1/n}$) is the reciprocal of the sum of the semi-axes of the largest ellipse with foci at ± 1 on which ϱ is holomorphic.

For the Fourier case see for example [Ka, p. 26]. $\square$

The supremum norm of any Fourier character e_n on $\mathbb{T}$ is equal to $2^{-1/2}$. In particular this bound is independent of n. If, however, we consider any fixed complex neighbourhood U of $\mathbb{T}$, it is no longer true that the supremum norms are uniformly bounded in n, since $||e_n||_{A_\infty(U)} = \exp(n\pi \sup_{z\in U} |Im(z)|)$. To obtain uniform bounds on the transfer operator norms $L(n,\xi)$ appearing in §6, we would like a better control of the norms of the e_n on complex domains. We therefore consider a decreasing sequence of complex domains U_n, chosen so that $||e_n||_{A_\infty(U_n)}$ is uniformly bounded in n. Clearly this is possible by ensuring that the vertical height of U_n is $O(\frac{1}{n})$.

Lemma 7.2. *Let $e_n(x) = 2^{-1/2}e^{\pi n i x}$, for $n \in \mathbb{Z}$. Let $\{f_n\}_{n=0}^\infty$ be the re-ordering of the e_n so that $f_{2n} = e_{-n}$ and $f_{2n+1} = e_n$.*

Let $U_0 \supset U_1 \supset \dots$ be a decreasing sequence of complex domains, such that U_n is admissible for the pair (T, f_n), and $\sup_{z\in U_n} |Im(z)| = O(\frac{1}{n})$.

Then for any $\delta > 0$, the sequence of values $L(n,\delta) = \max_{|t|\le\delta} ||\mathcal{L}_{t,n}||_{A_2(U_n)}$ is uniformly bounded independently of n.

Proof. As observed above, the restriction on the vertical height of U_n is enough to ensure that the supremum norms $||f_n||_{A_\infty(U_n)}$ are bounded independent of n, by $J > 0$ say.

The norms $||\mathcal{L}||_{A_2(U_n)}$ of the Perron-Frobenius operator, as an operator $A_2(U_n) \to A_2(U_n)$ are clearly decreasing in n, since $U_0 \supset U_1 \supset \dots$. In particular there exists $E > 0$ such that $||\mathcal{L}||_{A_2(U_n)} \le E$ for all $n \ge 0$.

If $|t| \le \delta$ then we have the estimate

$$\begin{aligned} ||\mathcal{L}_{t,n}||^2_{A_2(U_n)} &= \int_{U_n} \left| \sum_{i=1}^d e^{tf_n(z)} v(T_i z) \right|^2 dA(z) \\ &\le ||e^{tf_n}||_{A_\infty(U_n)} \int_{U_n} \left| \sum_{i=1}^d v(T_i z) \right|^2 dA(z) \\ &\le e^{\delta J} E^2. \quad \square \end{aligned}$$

The case of normalised Legendre polynomials is slightly more involved. Each Legendre polynomial P_n has supremum norm on I equal to 1. The maximum modulus of P_n is achieved at the two endpoints, where $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ (see [WW], §15.1). The normalised basis functions $f_n = (n+\frac{1}{2})^{1/2} P_n$ therefore have supremum norm $||f_n||_{\infty,I} = (n+\frac{1}{2})^{1/2}$, which in particular is *not* uniformly bounded in n. Consequently for any sequence of complex neighbourhoods U_n of I we will have $||f_n||_{A_\infty(U)} \ge (n+\frac{1}{2})^{1/2}$. Nevertheless we would like a uniform bound on operator norms analogous to that of Lemma 7.2. We are forced, therefore, to choose a sequence $\delta_n \searrow 0$, and prove that $L(n,\delta_n) = \max_{|t|\le\delta_n} ||\mathcal{L}_{t,n}||_{A_2(U_n)}$ can be bounded independently of n.

Lemma 7.3. *Let $\{f_n\}_{n=0}^{\infty}$ be the orthonormal basis for $L^2(I, dx)$ consisting of Legendre polynomials.*

Let $U_0 \supset U_1 \supset \ldots$ be a decreasing sequence of complex domains, such that U_n is admissible for the pair (T, f_n), and dist$(I, \partial U_n) = O(n^{-2})$.

Let δ_n be a real sequence decreasing to zero, with $\delta_n = O(n^{-1/2})$.

Then $L(n, \delta_n) = \max_{|t|\le\delta_n} ||\mathcal{L}_{t,n}||_{A_2(U_n)}$ is uniformly bounded independently of n.

Proof. The condition $\mathrm{dist}(I, \partial U_n) = O(n^{-2})$ ensures that the norms $||P_n||_{A_\infty(U_n)}$ of the Legendre polynomials are bounded independent of n, so that $||f_n||_{A_\infty(U_n)} = O((n+\frac{1}{2})^{1/2})$. The condition on the decay of δ_n then ensures that $\delta_n||f_n||_{A_\infty(U_n)}$ is bounded by some constant $F > 0$, say.

We can now argue as in Lemma 7.2. Firstly there is a uniform bound $||\mathcal{L}||_{A_2(U_n)} \le E$ on the norms of the Perron-Frobenius operator $\mathcal{L}$ on the spaces $A_2(U_n)$. Secondly, if $|t| \le \delta_n$ then

$$\begin{aligned}
||\mathcal{L}_{t,n}||^2_{A_2(U_n)} &= \int_{U_n} \left| \sum_{i=1}^{d} e^{tf_n(z)} v(T_i z)\right|^2 dA(z) \\
&\le ||e^{tf_n}||_{A_\infty(U_n)} \int_{U_n} \left| \sum_{i=1}^{d} v(T_i z)\right|^2 dA(z) \\
&\le e^F E^2. \quad \square
\end{aligned}$$

8. Proof of Theorems

We are now ready to combine all our previous estimates to prove Theorems 1 and 2.

Let $\{f_n\}_{n=0}^{\infty}$ be our orthonormal basis, consisting of either the Legendre polynomials or the (re-ordered) Fourier characters. If ϱ is our invariant density, then by Lemma 7.1 we have the bound $||\varrho - \sum_{n=0}^{N}(\varrho, f_n)f_n||_{\infty,I} \le \tilde{K}_\varrho \alpha_\varrho^N$ for constants $\tilde{K}_\varrho > 0$ and $0 < \alpha_\varrho < 1$.

Consequently,

$$\begin{aligned}
&\left|\left| \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n - \varrho \right|\right|_{\infty,I} \\
&\qquad \le \left|\left| \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n - \sum_{n=0}^{N} (\varrho, f_n) f_n \right|\right|_{\infty,I} + \left|\left| \sum_{n=0}^{N} (\varrho, f_n) f_n - \varrho \right|\right|_{\infty,I} \qquad (8.1) \\
&\qquad \le \left|\left| \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n - \sum_{n=0}^{N} (\varrho, f_n) f_n \right|\right|_{\infty,I} + \tilde{K}_\varrho \alpha_\varrho^N \ ,
\end{aligned}$$

so it remains to analyse the term $\left|\left|\sum_{n=0}^{N}(\varrho, f_n)^{(M)} f_n - \sum_{n=0}^{N}(\varrho, f_n) f_n\right|\right|_{\infty,I}$.

We clearly have

$$\begin{aligned}\left|\left|\sum_{n=0}^{N}(\varrho,f_n)^{(M)}f_n-\sum_{n=0}^{N}(\varrho,f_n)f_n\right|\right|_{\infty,I} &\le \sum_{n=0}^{N}||f_n||_{\infty,I}\left|(\varrho,f_n)^{(M)}-(\varrho,f_n)\right| \\ &\le \frac{1}{\kappa}\sum_{n=0}^{N}||f_n||_{\infty,I}\left(\frac{B(\theta_n)}{\xi}\left[\frac{\theta_n^{M(M+1)/2}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^M}{1-\left(\frac{L(n,\xi)}{1-\theta_n}\right)\theta_n^{M/2}}\right]+\lambda(M)K_\varrho\alpha_\varrho^n\right),\end{aligned} \tag{8.2}$$

the latter inequality holding for all $\xi > 0$ and all sufficiently large M (and hence all sufficiently large N) by Lemma 6.5, where $\kappa > 0$ is a constant, independent of both M and N.

We now estimate the disc contraction ratios θ_n. In the case of Fourier expansions, by Lemma 7.2 the vertical height of the sequence of complex domains $U_0 \supset U_1 \supset \dots$ must be bounded by $\frac{D}{n}$, for some fixed $D > 0$. Setting $\varepsilon_n = \frac{D}{n}$ we can choose $U_n = \varphi_{\varepsilon_n,L}(\mathbb{D})$, where the family of biholomorphisms $\varphi_{\varepsilon,L}$ was defined in §6. By Lemma 6.2 we therefore have $\theta_N \sim 1 - \exp(-qN)$ as $N \to \infty$, where $q = 2\tau LD$ (recall that $0 < \tau < 1$ is a contraction ratio valid for all the domains $U_{\varepsilon,L}$).

Choose $C > q$. Let us suppose that $M \sim \exp(CN)$, as in Theorem 2. We then calculate that

$$\begin{aligned}\frac{L(N,\xi)}{1-\theta_N}\theta_N^{M/2} &\sim L(N,\xi)e^{qN}\left(1-e^{-qN}\right)^{\exp(CN)/2} \\ &\sim L(N,\xi)e^{qN}(1-e^{-qN})^{\exp(qN)\exp([C-q]N)/2}, \\ &\sim L(N,\xi)e^{qN}(e^{-1/2})^{\exp([C-q]N)}\end{aligned}$$

which tends to zero as $N \to \infty$. In particular we have

$$\frac{L(N,\xi)}{1-\theta_N}\theta_N^{M/2} < \frac{1}{2} \quad \text{for sufficiently large } N.$$

For Legendre expansions we use Lemmas 6.2 and 7.3, together with an argument analogous to the one below for the Fourier case, to find $q > 0$ such that $\theta_N \sim 1 - \exp(-qN^2)$ as $N \to \infty$.

As in Theorem 1, we assume that $M \sim \exp(CN^2)$, where again $C > q$. A calculation analogous to the one above shows that

$$\frac{L(N,\xi)}{1-\theta_N}\theta_N^{M/2} \sim L(N,\xi)e^{qN^2}\left(1-e^{-qN^2}\right)^{\exp(CN^2)/2} \sim L(N,\xi)e^{qN^2}(e^{-1/2})^{\exp([C-q]N^2)},$$

which tends to zero as $N \to \infty$. In particular we again have that

$$\frac{L(N,\xi)}{1-\theta_N}\theta_N^{M/2} < \frac{1}{2} \quad \text{for sufficiently large } N.$$

Let δ_n be a non-increasing sequence of positive real numbers. In the case of normalised Legendre polynomials we assume $\delta_n = O(n^{-1/2})$ while in the case of Fourier characters we assume $\delta_n = \delta > 0$ is a constant sequence.

By Lemmas 7.2 and 7.3 we deduce that $L(n,\delta_n)$ is uniformly bounded in n. Now θ_n is increasing with n, so we can estimate

$$\frac{1}{1-\frac{L(n,\delta_n)}{1-\theta_n}\theta_n^{M/2}} < \frac{1}{1-\frac{L(N,\delta_N)}{1-\theta_N}\theta_N^{M/2}} < 1.$$

Setting $\xi=\delta_n$ in (8.2) we have

$$\begin{aligned}
&\left|\left|\sum_{n=0}^{N}(\varrho,f_n)^{(M)}f_n-\sum_{n=0}^{N}(\varrho,f_n)f_n\right|\right|_{\infty,I} \\
&\qquad\le \frac{1}{\kappa}\sum_{n=0}^{N}||f_n||_{\infty,I}\left(\frac{B(\theta_n)}{\delta_n}\left[\theta_n^{M(M+1)/2}\left(\frac{L(n,\xi)}{1-\theta_n}\right)^M\right]+\lambda(M)K_\varrho\alpha_\varrho^n\right) \\
&\qquad\le \frac{(N+1)||f_N||_{\infty,I}}{\kappa}\left(\frac{\lambda(M)K_\varrho}{1-\alpha_\varrho}+\frac{B(\theta_N)\theta_N^{M(M+1)/2}}{\delta_N}\left(\frac{Q}{1-\theta_N}\right)^M\right).
\end{aligned} \tag{8.3}$$

In the final inequality, $Q>0$ denotes the uniform bound for $L(n,\delta_n)$ which is guaranteed by Lemmas 7.2 and 7.3, and we have used the fact that both $1/\delta_n$ and $||f_n||_{\infty,I}$ are non-decreasing with n, while both θ_n and $B(\theta_n):=\prod_{m=1}^{\infty}(1-\theta_n^m)^{-1}$ are increasing with n.

We now wish to analyse more carefully the growth of $B(\theta_N)$, and are grateful to Oscar Bandtlow for showing us the following argument, establishing the exponential bound

$$B(\theta_N):=\prod_{m=1}^{\infty}(1-\theta_N^m)^{-1}\le \exp\left(\frac{-\pi^2}{6\log\theta_N}\right). \tag{8.4}$$

In particular, if $\theta_N\sim 1-e^{-qN}$ for $q>0$ then this gives

$$B(\theta_N)\le \exp\left(\frac{\pi^2}{6}e^{qN}\right), \tag{8.5a}$$

while if $\theta_N\sim 1-e^{-qN^2}$ then

$$B(\theta_N)\le \exp\left(\frac{\pi^2}{6}e^{qN^2}\right). \tag{8.5b}$$

To prove (8.4) we first recall that the *dilogarithm* is defined by the power series $\mathrm{Li}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$, and has the integral representation $\mathrm{Li}_2(x)=\int_x^0\frac{\log(1-s)}{s}\,ds$. Then

$$\sum_{m=1}^{k}-\log(1-\theta_N^m)\le\int_0^k-\log(1-\theta_N^t)\,dt=\int_0^k-\log(1-e^{-\gamma_N t})\,dt$$

where we write $\gamma_N=-\log\theta_N$ (>0).

Making the change of variable $s = e^{-\gamma_N t}$ we obtain

$$\int_0^k -\log(1-e^{-\gamma_N t})\,dt = \int_{s=1}^{\exp(-\gamma_N k)} \frac{\log(1-s)}{\gamma_N s}\,ds$$
$$= \frac{1}{\gamma_N}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(e^{-\gamma_N k})\right).$$

Letting $k \to \infty$ we obtain the estimate

$$\sum_{m=1}^{\infty} -\log(1-\theta_N^m) \le \gamma_N^{-1}(\mathrm{Li}_2(1) - \mathrm{Li}_2(0)) = \frac{-\pi^2}{6\log\theta_N},$$

and hence

$$B(\theta_N) = \prod_{m=1}^{\infty}(1-\theta_N^m)^{-1} = \exp - \sum_{m=1}^{\infty}\log(1-\theta_N^m) \le \exp\left(\frac{-\pi^2}{6\log\theta_N}\right),$$

which is the inequality (8.4). If $\theta_N = 1-x$ then $-\log\theta_N = \sum_{j=1}^{\infty}\frac{1}{j}x^j \ge x$, and (8.5a), (8.5b) follow.

We now express the righthand side of (8.3) as $R(N) = R_1(N) + R_2(N)$ (we suppress the dependence on M, since $M = M(N)$), where

$$R_1(N) = \frac{(N+1)||f_N||_{\infty,I}}{\kappa}\frac{\lambda(M)K_\varrho}{1-\alpha_\varrho}$$

and

$$R_2(N) = \frac{(N+1)||f_N||_{\infty,I}}{\kappa}\frac{B(\theta_N)\theta_N^{M(M+1)/2}}{\delta_N}\left(\frac{Q}{1-\theta_N}\right)^M.$$

We observe that $R_1(N)$ is converging rapidly to zero, due to the $\lambda(M)$ factor which, as noted in Lemma 5.4, is $O(\beta^{M^2})$ as $M \to \infty$, for some $0 < \beta < 1$. Since either $M \sim e^{CN}$ or $M \sim e^{CN^2}$ then $\lambda(M)$ is certainly $O(\gamma^N)$ as $N \to \infty$, for every $0 < \gamma < 1$. The other factor $\frac{(N+1)||f_N||_{\infty,I}}{\kappa}\frac{K_\varrho}{1-\alpha_\varrho}$ has comparatively slow growth; it is $O(N^{3/2})$ in the case of the normalised Legendre basis, and $O(N)$ in the case of the Fourier basis. Consequently

$$R_1(N) = O(\gamma^N) \text{ as } N \to \infty\,, \text{ for every } 0 < \gamma < 1\,. \tag{8.6}$$

To analyse $R_2(N)$ we write $R_2(N) = R_3(N)R_4(N)$, where

$$R_3(N) = \frac{(N+1)||f_N||_{\infty,I}}{\kappa\delta_N} = O(N^2)$$

(and in the Fourier case we have the improved bound $R_3(N) = O(N)$).

In particular we will now show that the modest growth of $R_3(N)$ is dominated by the rapid decay of

$$R_4(N) = B(\theta_N)\left(\frac{Q}{1-\theta_N}\right)^M \theta_N^{M(M+1)/2}.$$

It will be convenient to treat the Fourier and Legendre cases separately.

The case of Fourier series. Since $\theta_N \sim 1 - e^{-qN}$ and $M \sim \exp(CN)$ for a fixed $C > q$ then

$$\theta_N^{M(M+1)/2} \sim ((1 - e^{-qN})^{e^{qN}})^{e^{(C-q)N}/2} \sim (e^{-1})^{e^{(C-q)N}/2} = \exp\left(-e^{(C-q)N}/2\right).$$

Combining this with (8.5a) gives

$$\log R_4(N) \sim \frac{\pi^2 e^{qN}}{6} + M \log Q - qMN - e^{(C-q)N}/2.$$

Since $M \sim e^{CN}$ we see that the dominant term is $-qMN = -qe^{CN}N$ if $C \leq 2q$, and is $-e^{(C-q)N}/2$ if $C > 2q$, so in both cases $\log R_4(N) \to -\infty$ as $N \to \infty$. Therefore in either case we have

$$R_4(N) = O\left(\exp\left(-qe^{CN}N\right)\right),$$

which converges to zero faster than any exponential. Therefore for any $0 < \gamma < 1$ we have $R_2(N) = R_3(N)R_4(N) = O(\gamma^N)$ as $N \to \infty$.

The case of Legendre series. Since $\theta_N \sim 1 - e^{-qN^2}$ and $M \sim \exp(CN^2)$ for a fixed $C > q$ then

$$\theta_N^{M(M+1)/2} = ((1 - e^{-qN^2})^{e^{qN^2}})^{e^{(C-q)N^2}/2} \sim (e^{-1})^{e^{(C-q)N^2}/2} = \exp\left(-e^{(C-q)N^2}/2\right).$$

Combining this with (8.5a) gives

$$\log R_4(N) \sim \frac{\pi^2 M}{6} + M \log Q - qMN^2 - e^{(C-q)N^2}/2.$$

Setting $M = e^{CN^2}$ we see that if $C \leq 2q$ then the dominant term is $-qMN^2 = -qe^{CN^2}N^2$, while if $C > 2q$ then the dominant term is $-e^{(C-q)N^2}/2$. In both cases we see that $\log R_4(N) \to -\infty$ as $N \to \infty$, and that

$$R_4(N) = O\left(\exp\left(-qe^{CN^2}N^2\right)\right),$$

which converges to zero faster than any exponential. Therefore we have

$$R_2(N) = R_3(N)R_4(N) = O(\gamma^N) \text{ as } N \to \infty, \text{ for any } 0 < \gamma < 1. \tag{8.7}$$

So in both the Fourier and the Legendre cases, combining (8.6) and (8.7) allows us to deduce that

$$\begin{aligned}&\left\| \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n - \sum_{n=0}^{N} (\varrho, f_n) f_n \right\|_{I,\infty} \\ &\qquad \leq R(N) = R_1(N) + R_2(N) = O(\gamma^N) \quad \text{as } N \to \infty, \text{ for every } 0 < \gamma < 1.\end{aligned}$$

Combining the above estimate with (8.1) gives us

$$\left\| \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n - \varrho \right\|_{\infty,I} = O(\alpha_\varrho^N) \quad \text{as } N \to \infty,$$

thus completing the proof of Theorems 1 and 2. $\square$

9. C^k EXPANDING MAPS

Suppose that T is a topologically mixing piecewise expanding Markov map, but is only piecewise[5] C^k, for some $2 \leq k \leq \infty$, rather than piecewise real analytic. In this case there exists an absolutely continuous invariant probability measure, whose density function ϱ is C^{k-1}. The requirement $k \geq 2$ can be weakened; if the derivative of T is Hölder continuous, say, then again there exists an absolutely continuous invariant probability measure, though if T is merely C^1 then this need not be the case (cf. [GS]).

All of the previous formulae for approximating ϱ only involve the *first* derivative of T, so at least formally they make sense when T is C^k. Indeed the algorithm *does* produce a sequence of approximations to ϱ in this case, though in general their rate of convergence is slower than in the C^ω case.

The reasons for this are twofold. Firstly the expansion coefficients (ϱ, f_n) are only $O(\frac{1}{n^{k-1}})$ as $n \to \infty$. Secondly, the decay of the Taylor coefficients c_m and ${}_nc'_m(0)$ will only be exponential in n, so that $(\varrho, f_n)^{(M)}$ defined in Proposition 3.1 will converge to (ϱ, f_n) at an exponential, rather than superexponential, rate. This is because the determinants defined by (2.1), (2.2), (3.1) are not entire functions of z. Strictly speaking they are not determinants in the classical sense, since the corresponding transfer operators are not trace class (nor even compact, though they are quasi-compact, cf. [Ru4]). Nevertheless these determinants do have a meromorphic extension to a complex disc centred at the origin and containing the zero of minimum modulus (see [Ru4]), thus ensuring the exponential decay of their Taylor coefficients.

An alternative approach is to first approximate the C^k map T by a C^ω expanding map S. Such an approximation is possible in various topologies, in particular the Banach C^k topology; explicitly this can be achieved by convolving T with a suitable summability kernel (cf. [Kat]). One then works with the map S, applying the algorithm described in §§1–8 to construct an approximation $\tilde{\varrho}_S$ to the density ϱ_S of S. But ϱ_S is close to the density ϱ_T of T in various topologies, in particular the uniform topology, so $\tilde{\varrho}_S$ is a uniform approximation to ϱ_T, as required.

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