ENTROPY, EXPONENTS AND INVARIANT DENSITIES FOR HYPERBOLIC SYSTEMS: DEPENDENCE AND COMPUTATION

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ABSTRACT. In this survey we shall discuss related aspects of different invariants for hyperbolic systems. Firstly, we describe the dependence of the (topological) entropy on the underlying system. Secondly, we consider methods for efficient computation of (metric) entropy, and related quantities.

1. INTRODUCTION

Dynamical systems are often characterized in terms of simple quantities such as topological entropy, metric entropy, and Lyapunov exponents.

1.1. **Topological entropy.** Topological entropy is a fundamental invariant in topological dynamics. We can consider its dependence on the underlying system. For general $C^{1+\alpha}$ diffeomorphisms of compact surfaces we have the following well known result.

Theorem 1.
(1) *If M is a compact surface then for $C^{1+\alpha}$ diffeomorphisms on M the topological entropy map $T \mapsto h_{top}(T)$ is lower semi-continuous (Katok* [Ka1]*).*
(2) *If M is any compact manifold then for C^∞ diffeomorphisms on M the topological entropy map $T \mapsto h_{top}(T)$ is upper semi-continuous (Yomdin* [Yo]*, Newhouse* [Ne]*).*

In particular, for a compact surface M the map $T \mapsto h_{top}(T)$ is a continuous function of the C^∞ diffeomorphism $T : M \to M$.

There are corresponding results for, respectively, $C^{1+\alpha}$ flows on three dimensional manifolds [Ka1], and C^∞ flows on manifolds of arbitrary dimension. Katok's elegant result uses Pesin Theory to construct generalized hyperbolic horseshoes within the surface whose topological entropies are arbitrarily close to $h(f)$. Such horseshoes persist under small perturbations and thus the entropy cannot suddenly drop. In higher dimensions the analogous results may easily fail. This was shown by Misiurewicz [Mi] in dimensions ≥ 4. The approach of Yomdin uses ideas on semi-algebraic sets (see [Yo], [Gro]).

In the class of Anosov systems there is another interesting story. Of course, for Anosov diffeomorphisms structural stability trivially gives that the topological entropy is constant under perturbation, since it is a conjugacy invariant. However, in the case of Anosov flows, the topological entropy will be expected to vary under perturbations in the flow. However, it does so with suprising regularity.

Theorem 2.
(1) *For a C^1 perturbation of an Anosov flow the topological entropy varies C^1 (Katok, Knieper and Weiss* [KKW]*); and*
(2) *For a C^k ($k \geq 1$, $k = \omega$) perturbation of an Anosov flow the topological entropy varies C^{k-1} (Katok, Knieper, Pollicott and Weiss* [KKPW]*).*

There are also formulae for the first derivative [KKW] and higher derivatives [Po2], [Fl], which have proved useful in various contexts (cf. eg. [Pa^2]).

In the finite differentiability case the proof of these results follows from a careful study of the dependence of the conjugacy map on the perturbation, while in the C^ω case there is an alternative proof using dynamical zeta functions [KKPW].

1.2. **Metric entropy.** There are analogous results for the metric entropy of the SRB measure μ for Anosov systems.

Theorem 3.
(1) *For a C^k ($k \geq 2$) perturbation of an Anosov diffeomorphism the metric entropy varies C^{k-1}.*
(2) *For a C^k ($k \geq 2$) perturbation of an Anosov flow the topological entropy varies C^{k-1}. (Contreras [Co])*

As for Theorem 2, the proof of these results in the finite differentiability case also follows from a careful study of the dependence of the structural stability map [Co], while the C^ω case admits an alternative proof via zeta functions [Po2]. Again there are explicit formulae for the derivatives [Fl].

1.3. **Computation of entropy and densities.** For concrete hyperbolic systems it is a challenge to estimate dynamical characteristics such as entropy and Lyapunov exponents. In general the SRB measure is not expressible in closed form, and there is no simple expression for either its metric entropy or its Lyapunov exponents, so there is interest in practical means of approximating these quantities. If the SRB measure is absolutely continuous with respect to Lebesgue measure then it is also desirable to approximate the corresponding density function.

For hyperbolic diffeomorphisms and flows there are few rigorous results on the approximation of these quantities. In the model case of expanding interval maps, however, this is a well studied problem. Such maps always have an invariant measure μ which is absolutely continuous with respect to Lebesgue measure, and here we shall survey various methods of approximating μ itself as well as its entropy. The principal methods include

(1) The computation of long time averages,
(2) Ulam's method of phase space discretisation,
(3) The finite section method,
(4) Iteration of the Perron-Frobenius operator,
(5) Periodic orbits and Fredholm determinants of transfer operators.

This article is organised as follows. In §2 we consider the entropy of geodesic flows on spaces of negative curvature, while §3 treats the entropy of more general Anosov flows. In §4 we outline various approaches to the computation of metric entropy for expanding maps. The variation of this entropy, and of the corresponding invariant density, are discussed in §5. In §6 this is related to the approximation of metric entropy for C^k expanding maps using periodic points. Finally in §7 this same method is applied to the computation of entropy of Anosov flows.

2. An application to geometry: Geodesic flows

Before embarking on a detailed discussion of results on the regularity and computation of entropy, we shall give an interesting special case where there are some particularly elegant results.

For the special case of geodesic flows on negatively curved manifolds there is interest in locating the critical points of the entropy map, and expressing its derivatives in terms of geometric data. This has connections with entropy minimizing metrics of fixed volume. Let $\mathcal{M}$ be the space of C^∞ metrics g on a compact manifold M which have strictly negative sectional curvatures and such that the associated volume is normalized to $\mathrm{Vol}_g(M) = 1$. The geodesic flow associated to $g \in \mathcal{M}$ is Anosov. Given $g \in \mathcal{M}$, let $h(g)$ denote the entropy of the geodesic flow on M with respect to g. The following is a nice geometric interpretation due to Manning [Man1].

Proposition 4. *Let $\widetilde{M}$ denote the universal cover of M and let $\tilde{g}$ denote the lift of the metric g to $\widetilde{M}$. We can identify*

$$h(g) = \lim_{T \to +\infty} \frac{1}{T} \log \mathrm{Vol}_{g_0}\{x : d_g(x_0, x) \leq T\}$$

for any $x_0 \in \widetilde{M}$.

Let g_0 be a metric with constant sectional curvatures. The following result shows that entropy plays an important role in characterizing metrics.

Proposition 5.
(1) *If M is a surface then for any $g \in \mathcal{M}$ we have that $h(g) \geq h(g_0)$ (Katok* [Ka1]*).*
(2) *If M has dimension at least* 3 *then for any $g \in \mathcal{M}$ we have that $h(g) \geq h(g_0)$, with equality iff $g = g_0$ (Besson, Courtois and Gallot* [BCG1], [BCG2]*)*

Part (1) of this Proposition is a result of Katok [Ka2] from 1982. Since the surface supports a metric of negative curvature, there is a finite dimensional space of metrics of constant curvature with entropy $h(g_0)$, corresponding to the Teichmuller space of the surface. Part (2), often referred to as the Gromov-Katok conjecture, was eventually proved in 1991 [BCG1]. Mostow rigidity ensures that in (2) there is a unique constant curvature metric g_0, which is the unique minimum for the entropy.

The proofs of Proposition 5 do not depend on the smoothness of $h : \mathcal{M} \to \mathbb{R}$. However, by Theorem 2 the entropy map $h(g)$ on $\mathcal{M}$ is C^∞. In particular, the critical points occur precisely when $Dh(g) = 0$.

Consider a parametrized family of metrics $g^\lambda = g_0 + \lambda g_1 + O(\lambda^2)$, for $\lambda \in (-\epsilon, \epsilon)$, say. Katok, Knieper and Weiss identified $Dh(g_0) = \int g_1(v, v) d\mu(v)$, where μ is the measure of maximal entropy (or Bowen-Margulis measure). Since $\mathrm{Vol}(g^\lambda) = 1$ we have that

$$D\mathrm{Vol}_g(g_0) = -\frac{h(g_0)}{2} \int g_1(v, v) d(\mathrm{Liou})_g = 0\,, \tag{1}$$

where $(\mathrm{Liou})_g$ is the Liouville measure [Be] and thus critical points are characterized by the equality of Bowen-Margulis measure and volume. In particular, at metrics g_0 with constant sectional curvatures we have that $\mu = \mathrm{Vol}_{g_0}$ and so $h(g)$ has a critical point. A local analysis of the second derivative gives some local results, but the global result of Besson et al uses a fundamentally different approach based on volume and barycentres [BCG1], [BCG2]). The derivative formula (1) has recently been used by Manning [Man2] to show that, for compact surfaces of negative curvature, the entropy $h(g_t)$ is strictly decreasing along any path $t \mapsto g_t$ of the Ricci flow.

We can also associate to metrics $g \in \mathcal{M}$ the metric entropy $h_{\mathrm{met}}(g)$. By Theorem 3 we can deduce that $h_{\mathrm{met}}(g)$ has a C^∞ dependence on $g \in \mathcal{M}$. By a careful analysis of the second derivatives one can show that the behaviour of $g \mapsto h_{\mathrm{met}}(g)$ is quite subtle. The following striking local result (due to Flaminio [Fl]) shows that the behaviour of metric entropy is very different to that of the topological entropy described in Proposition 5.

Proposition 6. *There exist examples of manifolds M with a metric g_0 of constant curvature such that:*
(a) *there exist metrics $g \in \mathcal{M}$ arbitrarily close to g_0 such that $h_{met}(g) > h_{met}(g_0)$, and*
(b) *there exist metrics $g' \in \mathcal{M}$ arbitrarily close to g_0 such that $h_{met}(g') < h_{met}(g_0)$.*

3. Regularity of entropy for Anosov systems: Methodology

Theorem 1 and Proposition 5 were both introduced for historical context. Their elegant proofs, the former involving Pesin Theory, the latter being more geometric in nature, are beyond the scope of these notes. Instead we shall concentrate on the ideas behind the proofs of Theorems 2 and 3, which will to some extent be echoed in section §5 when we consider the smooth variation of densities of expanding maps.

3.1. C^1 **topological entropy for Anosov flows.** The original approach of Katok, Knieper and Weiss used the definition of topological entropy directly, together with the even distribution of closed orbits with respect to the measure of maximal entropy. Let $\phi^{(\lambda)}$, $\lambda \in (-\epsilon, \epsilon)$ be a C^1 family of Anosov flows. Structural stability gives for each λ:

(1) Hölder homeomorphisms $\pi^{(\lambda)} : M \to M$ which map $\phi^{(0)}$ orbits to $\phi^{(\lambda)}$ orbits;
(2) C^0 reparameterizations $\rho^{(\lambda)} : M \to \mathbb{R}^+$ such that the reparameterization of $\phi^{(0)}$ by $\rho^{(\lambda)}$ is topologically conjugate to $\phi^{(\lambda)}$.

By considering lengths of closed orbits one sees that

$$h(\phi^{(0)}) \left(\lambda \int \left[\frac{\partial \rho^{(t)}}{\partial t}|_{t=0} \right] d\mu^{(0)} + o(\lambda) \right) \leq h(\phi^{(\lambda)}) - h(\phi^{(0)})$$
$$\leq h(\phi^{(0)}) \left(\lambda \int \left[\frac{\partial \rho^{(t)}}{\partial t}|_{t=0} \right] d\mu^{(\lambda)} + o(\lambda) \right) ,$$

where $\mu^{(\lambda)}$ is the Bowen-Margulis measure for $\phi^{(\lambda)}$. However, the upper semi-continuity of the measure theoretic entropy $\mu \mapsto h_\mu(\phi)$ implies that $\mu^{(\lambda)}$ converges to $\mu^{(0)}$ in the weak* topology. Thus we not only see that $h(\phi^{(\lambda)})$ is C^1 at $\lambda = 0$, but also with derivative $\int \left[\frac{\partial \rho^{(t)}}{\partial t}|_{t=0} \right] d\mu^{(0)}$.

3.2. C^k **topological entropy for Anosov flows.** Two main ingredients in the case of C^k flows ($k \geq 2$) are the structural stability of Anosov flows and symbolic dynamics. This then leads to a characterization of entropy in terms of thermodyanmic pressure and exploits its regularity.

We recall elements of the proof of structural stability for a C^k family of Anosov flows $\phi^{(\lambda)}$, $\lambda \in (-\epsilon, \epsilon)$, following the classical approaches of Mather [Mat] and Moser [Mo] (cf. also [LMM]). The vector field associated to $\phi^{(\lambda)}$ can be denoted $\Phi^{(\lambda)}$. In this context, structural stability gives

(1) a C^{k-1} family of Hölder homeomorphisms $\pi^{(\lambda)} : M \to M$ which map $\phi^{(0)}$ orbits to $\phi^{(\lambda)}$ orbits;
(2) a C^{k-1} family of Hölder reparameterizations $\rho^{(\lambda)} : M \to \mathbb{R}^+$ such that the reparameterization of $\phi^{(0)}$ by $\rho^{(\lambda)}$ is topologically conjugate to $\phi^{(\lambda)}$.

Here differentiability is interpreted in terms of functions with values in the Banach space $C^\alpha(M, \mathbb{R})$ or the Banach manifold $C^\alpha(M, M)$, and which are also differentiable in the flow direction. Crucially, all of these quantities are solutions to the identity

$$F(\Phi^{(\lambda)}, \pi^{(\lambda)}, \rho^{(\lambda)}) := \Phi^{(\lambda)} \circ \pi^{(\lambda)} - \rho^{(\lambda)} D\pi^{(\lambda)} \Phi^{(0)} = 0.$$

By the implicit function theorem one can solve for $\pi^{(\lambda)}$ and $\rho^{(\lambda)}$, where $\pi^{(0)} = id$ and $\rho^{(0)} = 1$, which moreover automatically yields the regularity of the solutions. The main ingredient is the regularity of the composition map.

Regularity of Composition Operators *Let M be a compact manifold. If $k \geq 2$ and $0 < \alpha < 1$, then the composition map $Comp(f, g) = f \circ g$ satisfies*

$$\text{Comp} : C^k(M, \mathbb{R}) \times C^0(M, M) \to C^0(M, \mathbb{R}) \text{ is } C^k.$$
$$\text{Comp} : C^k(M, \mathbb{R}) \times C^\alpha(M, M) \to C^\alpha(M, \mathbb{R}) \text{ is } C^{k-1}.$$

The first of these statements is covered by a well known unpublished preprint of Franks [Fr] (see also [Ir]), while a more recent comprehensive treatment by de la Llave-Obaya [LO] gives an excellent reference for both parts.

By a consideration of the regularity of composition we see that $F(\cdot, \cdot, \cdot)$ is C^{k-2}. The application of the implicit function theorem depends on the invertibility of $D_{2,3}F(\Phi^{(0)}, Id, 1)$, which is a simple consequence of the hyperbolicity of the flow. The Hölder exponent $0 < \alpha \leq 1$ is related to the regularity of the horocycle foliations.

A second ingredient is the use of symbolic dynamics. Using the original flow $\phi_t^{(0)} : M \to M$ we can construct a subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ and a Hölder continuous roof function $r : \Sigma_A \to \mathbb{R}$. The subshift Σ_A is associated to a finite number of sections $\mathcal{T} := \{T_1, \ldots, T_k\}$. The function $r : \Sigma_A \to \mathbb{R}$ corresponds to the transition function between sections. The sections $\mathcal{T}^{(\lambda)} := \{\pi^{(\lambda)}T_1, \ldots, \pi^{(\lambda)}T_k\}$ for the flow $\phi^{(\lambda)}$ give rise to precisely the same subshift Σ_A. However, there is a new roof function $r^{(\lambda)} : \Sigma_A \to \mathbb{R}$ defined by

$$r^{(\lambda)}(x) = \int_0^{r(x)} \rho^{(\lambda)}(\phi_t p(x))dt\,,$$

where $p : \Sigma_A \to \cup_i T_i$ is the coding map. In particular, the characterization of the topological entropy $h(\phi^{(\lambda)})$ in terms of the symbolic dynamics and the pressure function is given by $P(-h(\phi^{(\lambda)})r^{(\lambda)}) = 0$. This is a simple consequence of Abramov's Theorem on entropy and the variational principle for pressure. The pressure function is C^ω, which is a consequence of its characterization in terms of the Ruelle transfer operator and perturbation theory . Since $\frac{P(-tr^{(0)})}{dt}|_{t=h(\phi^{(0)})} \neq 0$, a second (trivial) application of the implicit function theorem shows that $\lambda \mapsto h(\phi^{(\lambda)})$ is C^{k-2}.

3.3. C^ω **topological entropy for Anosov flows.** Let τ be one of the countably many closed orbits for a C^ω Anosov flow ϕ and let $\lambda(\tau)$ be the associated least period. We can then associate the *dynamical zeta function* defined by

$$\zeta(s, \phi) := \prod_\tau \left(1 - e^{-s\lambda(\tau)}\right)^{-1}\,,$$

which defines a non-zero analytic function for $Re(s) > h(\phi)$.

Poles for ζ-functions *The zeta function $\zeta(s, \phi)$ has a meromorphic extension to a neighbourhood of $h(\phi)$, and $h(\phi)$ is a simple pole of $\zeta(s, \phi)$.*

This result appears in [PP], for example, but was originally proved by Ruelle as an exercise in his famous book "Theormodynamic formalism" [Ru2], (cf. [Ru3] for an amusing account). In [KKPW] there is a different approach to proving the smoothness of $\lambda \mapsto h(\phi^{(\lambda)})$ in the case of C^ω families of analytic functions using zeta functions, but it is limited to the analytic case. The basic idea is therefore that $h(\phi^{(\lambda)})$ occurs as a simple zero for the complex function $\zeta(s, \phi^{(\lambda)})^{-1}$. However, although the analytic extension of the function $\zeta(s, \phi^{(\lambda)})^{-1}$ to a neighbourhood of $h(\phi^{(\lambda)})$ uses symbolic dynamics [PP] the actual application of this result to the analyticity of the entropy depends instead on the theory of (several) complex variables. More precisely, one first considers an individual closed orbit τ and observes that the length $\lambda(\tau^{(\lambda)})$ of the associated closed orbit for $\phi^{(\lambda)}$ varies analytically. The first observation is that this dependence can be made uniform over all closed orbits. This leads to the conclusion that the function $\zeta(s, \phi^{(\lambda)})^{-1}$ has an analytic dependence on λ, at least on the part of the s-plane on which the zeta function converges. The second ingredient is that this analytic dependence can be extended to a neighbourhood of $h(\phi^{(\lambda)})$ by Hartog's theorem on two complex variables. Finally, an application of the ever useful implicit function theorem shows that $\lambda \mapsto h(\phi^{(\lambda)})$ is C^ω.

3.4. C^k **metric entropy for Anosov systems.** After the work in [KKPW] was complete Contreras, at the time a graduate student of Mañé, adapted the symbolic dynamics approach to show the regularity of metric entropy, both for Anosov diffeomorphisms and Anosov flows. The main new ingredients were ideas of Mañé, originally designed to describe the regularity of the Hausdorff dimension of hyperbolic Smale horsehoes. Naturally, it is also possible to adapt the zeta function approach to show C^ω regularity of the metric entropy [Po1].

4. COMPUTING METRIC ENTROPY FOR INTERVAL MAPS

We now turn to the model case of expanding interval maps. Let $\mathcal{I} = \{I_i\}_{i=1}^d$ be a partition of the interval I into subintervals I_i. Given $T : I \to I$, we shall always assume that $T|_{I_i}$ is C^k $(k \geq 2)$, for each i, and that T is *piecewise expanding* in the sense that there exists $\lambda > 1$ such that $|T'|\,|_{I_i} \geq \lambda$ for each i. We say that T is *Markov* if there exists a $d \times d$ matrix A (the *transition matrix* for T) with each entry either 0 or 1, such that $\mathrm{int}\,(T(I_i)) = \mathrm{int}\,\big(\cup_{j:A(i,j)=1} I_j\big)$ for each i, in which case $\mathcal{I}$ is called a *Markov partition* for T. Of course this Markov partition for T is not unique, as any refinement $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{I}$ is also a Markov partition. The map T is (topologically) mixing if some power of the transition matrix A is strictly positive. For each i, let $T_i : TI_i \to I_i$ denote the local inverse branch, with the property that $T_i \circ T = id|_{I_i}$. If $TI_i = I$ for every i, we say that T is a *full branch* expanding map; in this case each inverse branch T_i is defined on the whole of I. Although all results described in this section are true in the general context of mixing piecewise expanding Markov maps, we shall often make the assumption that T is full branch in order to simplify the exposition.

A classical result of Lasota and Yorke [LY] is:

Theorem 7. *If $T : I \to I$ is a mixing piecewise C^2 expanding Markov map, then there exists a unique ergodic T-invariant probability measure μ. The measure μ is T-exact, and has a Bernoulli natural extension. In particular, $h(\mu) > 0$.*

If T is C^k on each partition piece, for $k \geq 2$, then the density function ϱ, satisfying $d\mu(x) = \varrho(x)dx$, is C^{k-1} on each partition piece. If T is piecewise C^ω then so is ϱ.

If T is a full branch expanding map then ϱ is C^{k-1} on the whole of I whenever T is piecewise C^k, for $k = 2, 3, \ldots, \omega$.

We are concerned here with efficient methods of computing numerically the metric entropy $h(\mu)$ of the absolutely continuous invariant measure μ. Only in very special cases is an exact formula for $h(\mu)$ known, the most pertinent being the class of β-transformations $T_\beta(x) = \beta x \pmod 1$, where $h(\beta) = \log \beta$ [Pa]. An obvious first approach is to work directly from the definition of entropy. If $\mathcal{J} = \{J_i\}$ is a countable Borel partition of I then let

$$H(\mu, \mathcal{J}) = -\sum_i \mu(J_i) \log \mu(J_i) \tag{2}$$

denote its entropy, and define

$$h(\mu, \mathcal{J}) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{J}\right) . \tag{3}$$

Now $h(\mu) = \sup_{\mathcal{J}} h(\mu, \mathcal{J})$, though if $\mathcal{J}$ generates the Borel σ-algebra of I then $h(\mu) = h(\mathcal{J}, \mu)$ (cf. Thm. 4.18 of [Wa]). For a piecewise expanding Markov map, the initial Markov partition $\mathcal{I} = \{I_i\}_{i=1}^d$ is a generating partition, so that

$$h(\mu) = h(\mathcal{I}, \mu) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{I}\right) . \tag{4}$$

Unfortunately, in general there are several obstacles preventing a direct calculation of the right-hand side of (4). Firstly, except in simple cases the limit itself is not computable, so we must be content with an evaluation of $\frac{1}{n}H(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{I})$ for some large value of n. Secondly, it is clear that considerable computation is involved in identifying the elements of the refined partitions $\bigvee_{j=0}^{n-1} T^{-j}\mathcal{I}$, the computational effort being exponential in n. Thirdly, and most importantly, the measure μ is usually not available to us in closed form, so that an exact calculation of the entropy $H(\mu, \mathcal{J})$ of even a small finite partition $\mathcal{J}$ is problematic. This suggests that an immediate priority is the derivation of an approximation to the absolutely continuous invariant measure μ, in the form of either a weak* estimate or an approximation to its density function ϱ. Indeed an approximation of ϱ is the first step in almost all of the known techniques for computing metric entropy (see in particular subsections 4.2, 4.3 below). It is noteworthy, however, that probably the most efficient means of computing entropy, the *periodic point method* described in §4.5, works by circumventing any a priori computation of ϱ. Although this method *can* be used to build up an approximation to ϱ, by computing its Fourier coefficients for example (see §6), it is actually more efficient to *directly* compute $h(\mu)$.

It is doubtless possible to obtain reasonable estimates on $h(\mu)$ by working directly with (4) together with an approximation to μ. This approach is probably best used in conjunction with Ulam's method for approximating ϱ (see §4.2), which is explicitly couched in terms of refined partitions of I. However it is not clear whether a direct appeal to (4) is particularly efficient, nor whether rigorous error bounds can be obtained for any computed approximation.

It seems preferable to instead exploit the smooth structure of our dynamical system, in the form of the following classical Rohlin Lemma (cf. [Ro], [Pe]) equating the entropy of μ with its Lyapunov exponent.

Proposition 8. *("Rohlin Lemma") The metric entropy is equal to the Lyapunov exponent for μ, i.e.,*

$$h(\mu) = \int \log|T'|d\mu = \int \log|T'(x)|\rho(x)dx. \tag{5}$$

Rohlin's characterisation of entropy as a space average suggests a first method for its computation.

4.1. **Computing metric entropy: Time averages.** As previously noted, the absolutely continuous invariant measure μ is ergodic, so by Birkhoff's ergodic theorem we can equate (5) with the time average for $\log|T'|$:

$$h(\mu) = \int \log|T'|d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log|T'(T^i x)| \tag{6}$$

for Lebesgue almost every point x. To approximate $h(\mu)$ one simply computes finite averages $h_N(x) = \frac{1}{N}\sum_{i=0}^{N-1}\log|T'(T^i x)|$ for large values of N and particular choices of x. Although this approximation scheme is rather easy to implement, there are several drawbacks. Firstly, (6) is certainly not true for *every* $x \in I$, so care is needed in selecting x to be a Lebesgue-generic point for the map T. Secondly, in general an orbit segment $\{T^i(x)\}_{i=0}^{N-1}$ cannot be determined exactly by a computer. Since T is expanding, care is needed in avoiding round-off errors when computing approximate orbit segments. Both of these problems can in practice (though non-rigorously) be overcome by injecting some random noise, i.e. regularly replacing points $T^i(x)$ by randomly chosen nearby points y and thereafter computing iterates of y until the next injection of noise. Suitably executed, the resulting pseudo-orbit is more stable from a statistical point of view, as well as being immune to any round-off error.

A third drawback of computing $h(\mu)$ via time averages is that the ergodic theorem gives no bounds on the speed of convergence in (6), making it difficult to rigorously justify the quality of a computed approximation $h_N(x)$. If one is content with convincing, rather than rigorous, approximations to $h(\mu)$, then this problem is not particularly serious. In practice one would look for a diminished level of oscillation in the sequence $\{h_N(x)\}_N$, and make an empirical estimate of $h(\mu)$ with a corresponding level of confidence. A more rigorous approach to justifying the difference between a computed $h_N(x)$ and the limit (3) might be achieved via the central limit theorem, though there are undoubted difficulties in converting this into concrete error bounds[1].

From a philosophical and historical perspective, the notion of computing space averages such as (5) by means of time averages is a rather contrary one. Boltzmann's ergodic hypothesis was formulated in the hope of simplifying the work of the experimental physicist, by equating potentially slowly convergent time averages (the object of real interest) with space averages, the expectation being that the latter are more computationally accessible. The principal class of systems motivating this hypothesis were Hamiltonian, for which the relevant measure, Liouville measure, is known explicitly. Although applicable to more general, and potentially exotic, measures, the premise underlying Birkhoff's ergodic theorem, an appropriately modified version of Boltzmann's hypothesis, is nonetheless that space averages are easier to compute than time averages.

The systems we consider here are in general not Hamiltonian, and the relevant measure μ is rarely known exactly, but by no means is it exotic. The fact that it is absolutely continuous rather than singular suggests we should exploit this property and attempt to compute its Radon-Nikodym derivative ϱ. With an approximation $\tilde{\varrho}$ in hand, preferably with guaranteed error bounds, it is a comparatively simple task to evaluate the Riemann integral $\int \log|T'(x)|\,\tilde{\varrho}(x)\,dx$, either numerically or exactly, thereby providing an approximation to $h(\mu)$ by virtue of Rohlin's Lemma.

A straightforward means of approximating ϱ is to fix some partition of I, into subintervals J_i, say, and compute the proportion of time $p_i = \lim_N \frac{1}{N}\{0 \le j \le N-1 : T^j(x) \in J_i\}$ spent by a typical orbit in each J_i. The simple function $\sum_i p_i \chi_{J_i}$ is then an approximation to ϱ. Of course this method is essentially the same as in §4.1, and shares all of its drawbacks.

More satisfactory ways of computing ϱ are based on a functional analytic approach. For any continuous function $f : I \to \mathbb{R}$, the corresponding *Ruelle transfer operator* is defined by

$$\mathcal{L}_f v(x) = \sum_{Ty=x} e^{f(y)} v(y)\,,$$

[1]There are known estimates on the discrepancy of *typical* Birkhoff averages; eg. if T mixes at an exponential rate then the error is $O(n^{-\alpha})$, for any $0 < \alpha < \frac{1}{2}$ cf. Theorem 16 in [Kach].

the summation being over all preimages y of x. This defines a bounded linear operator on $C^0_{\mathcal{I}}(I)$, the Banach space of functions which are piecewise continuous with respect to the Markov partition $\mathcal{I} = \{I_i\}_{i=1}^d$. A particularly important special case corresponds to the choice $f = -\log|T'|$. The corresponding operator $\mathcal{L} = \mathcal{L}_{-\log|T'|}$ is the *Perron-Frobenius operator*, given explicitly by

$$\mathcal{L}v(x) = \sum_{Ty=x} \frac{v(y)}{|T'(y)|} = \sum_{i=1}^{d} v(T_i x)|T_i'(x)|\chi_{TI_i}(x)\,. \tag{7}$$

General transfer operators $\mathcal{L}_f$ were introduced by Ruelle, and are of central importance in his thermodynamic formalism [Ru2]. The Perron-Frobenius operator is a more classical construction, its importance stemming from the following observation, whose proof follows readily from the change of variables formula for integration.

Proposition 9. *The density function ϱ is the unique fixed point of the Perron-Frobenius operator $\mathcal{L} : C^0_{\mathcal{I}}(X) \to C^0_{\mathcal{I}}(X)$ which satisfies $\int \varrho(x)\,dx = 1$.*

If T is full branch then all Ruelle transfer operators, and in particular the Perron-Frobenius operator, in fact preserve $C^0(I)$, the space of continuous functions on I. If T is piecewise C^k for $k = 2, 3, \dots, \omega$ then $\mathcal{L}$ preserves $C^{k-1}_{\mathcal{I}}(I)$, and in the full branch case it preserves $C^{k-1}(I)$. Proposition 9 asserts in particular that 1 is a simple eigenvalue for $\mathcal{L}$. If $\mathcal{L}$ is viewed as acting on certain subspaces of $C^0_{\mathcal{I}}(I)$ then rather more is true: 1 is an *isolated* eigenvalue of $\mathcal{L}$. For example we have:

Proposition 10. *When acting on the space $C^r_{\mathcal{I}}(I)$, for any $1 \le r \le k-1$, the Perron-Frobenius operator $\mathcal{L}$ has spectral radius 1, and 1 is the only eigenvalue of maximum modulus. There exists $0 < \gamma < 1$ such that all other spectral values of $\mathcal{L} : C^r_{\mathcal{I}}(I) \to C^r_{\mathcal{I}}(I)$ have modulus less than γ.*

This result suggests the possibility of using some kind of spectral approximation to compute the leading eigenvector ϱ of $\mathcal{L}$. That is, one would explicitly compute the leading eigenvector of a finite rank operator $\tilde{\mathcal{L}}$ which is close to $\mathcal{L}$ in some suitable sense. In subsections 4.2 and 4.3 we describe two rather different finite rank approximation schemes.

4.2. **Computing metric entropy: Ulam's method.** This is the classical method for approximating the density function ϱ, first proposed by Ulam in 1960 [Ulam]. It is rather robust, in the sense that it can be applied to a wide class of maps $T : X \to X$ (see below for more details), so we first describe it in a general setting. Let $\mathcal{J} = \{J_i\}_{i=1}^l$ be a finite Borel partition of the compact metric space X, and let m be a probability measure on X. Define a projection operator $P_{\mathcal{J}}$ on $L^1(m)$ by

$$P_{\mathcal{J}} w = \sum_{i=1}^{l} \frac{\int_{J_i} w\,dm}{m(J_i)} \chi_{J_i}\,,$$

so that functions in the range of $P_{\mathcal{J}}$ are all locally constant with respect to the finite partition $\mathcal{J}$. The corresponding *Ulam approximant* is then defined to be the operator

$$\mathcal{L}_{\mathcal{J}} := P_{\mathcal{J}} \mathcal{L} P_{\mathcal{J}}\,.$$

Although $\mathcal{L}_{\mathcal{J}}$ preserves the space $C^k_{\mathcal{J}}(X)$ of functions which are piecewise C^k with respect to $\mathcal{J}$, in practice we often consider a sequence $\mathcal{J}_n$ of increasingly fine partitions, thereby necessitating a change of Banach space at each step. Specialising to the case $X = I$ it is often more convenient to view $\mathcal{L}_{\mathcal{J}}$ as an operator on BV, the space of functions of bounded variation. The finite dimensional space $V_{\mathcal{J}}$, consisting of functions which are piecewise constant with respect to $\mathcal{J}$, is invariant under $\mathcal{L}_{\mathcal{J}}$. Although the invariant density ϱ typically does not belong to $V_{\mathcal{J}}$, it can be shown (cf. [Liv]) that in order to approximate ϱ (or more generally all eigendata of $\mathcal{L}$ corresponding to *isolated* eigenvalues) it suffices to consider $\mathcal{L}_{\mathcal{J}}|_{V_{\mathcal{J}}}$. The operator $L_{\mathcal{J}} := \mathcal{L}_{\mathcal{J}}|_{V_{\mathcal{J}}}$ is represented by a finite matrix $M_{\mathcal{J}}$ of size $l \times l$, whose $(i,j)^{th}$ entry is

$$M_{\mathcal{J}}(i,j) = \frac{m(T^{-1}J_j \cap J_i)}{m(J_i)} \,.$$

Thus $M_{\mathcal{J}}$ can be interpreted as the transition matrix for a Markov chain, with $M_{\mathcal{J}}(i,j)$ representing the probability that a point in J_i is mapped into J_j by T.

The eigendata of $L_{\mathcal{J}}$ is therefore potentially explicitly computable. For this to be the case, both the measure m and the partition pieces J_i need to be sufficiently tractable. The measure m is commonly chosen to be Lebesgue measure, or alternatively to be atomic, with one atom in each partition piece and all atoms of equal weight. For a piecewise expanding Markov interval map, a natural choice of partition $\mathcal{J}$ is a refinement $\bigvee_{r=0}^{n-1} T^{-r}\mathcal{I}$ of the initial Markov partition $\mathcal{I}$.

Since $M_{\mathcal{J}}$ is a probability matrix, its leading eigenvalue is 1. Let $\varrho_{\mathcal{J}}$ denote the corresponding left eigenvector. Then $\varrho_{\mathcal{J}}$ is the leading eigenvector of $L_{\mathcal{J}}$, the *Ulam approximant* to the invariant density ϱ of T. The expectation is that $\varrho_{\mathcal{J}}$ will approximate ϱ as the partition $\mathcal{J}$ is refined. More precisely,

Theorem 11.

$$||\varrho - \varrho_{\mathcal{J}}||_{BV} = O(\varepsilon \log 1/\varepsilon) \quad \textit{as } \varepsilon \to 0\,,$$

where $\varepsilon = \varepsilon(\mathcal{J}) = \max_{1\le i\le l} \operatorname{diam}(J_i)$ *is the mesh size of* $\mathcal{J}$.

A proof of this result, which in fact remains valid for non-Markov piecewise-expanding maps, can be found in [KMY] and [Liv].

In the case of Markov expanding maps the mesh size of the refined Markov partition $\mathcal{J}_n := \bigvee_{r=0}^{n-1} T^{-r}\mathcal{I}$ is $O(\alpha^n)$ for some $0 < \alpha < 1$, so we have:

Theorem 12.

$$||\varrho - \varrho_{\mathcal{J}_n}||_{BV} = O(\alpha^n) \quad \textit{as } n \to \infty\,.$$

An appealing feature of Ulam's method is that it can be implemented for completely general maps T. It is therefore natural to speculate about whether its convergence properties can be extended beyond the class of purely expanding maps, perhaps to other uniformly hyperbolic systems. In this respect there is diverse evidence. On the one hand Froyland [Fro1] has shown that for 2-dimensional Anosov diffeomorphisms, if the partitions are chosen to be refinements of some initial Markov partition then a suitably modified Ulam scheme[2] does yield a weak* approximation to the SRB measure. Moreover, various numerical work (see for example [DJ], [Fro2]) suggests the convergence of the Ulam scheme for more general classes of systems. However, Blank, Keller & Liverani [BKL] have shown that if T is a hyperbolic automorphism of the 2-torus then the Ulam approximant $\mathcal{L}_{\mathcal{J}}$ can yield spurious spectral data for[3] $\mathcal{L}$ even when $\mathcal{J}$ is the most natural partition of $\mathbb{T}^2$ into squares of equal (and arbitrarily small) size. On the other hand they note that the approximation of the SRB measure by the Ulam scheme may still be possible for arbitrary partitions, in the sense that their investigations tend to confirm rather than contradict this possibility.

In the case of Markov dynamics which is piecewise smooth or piecewise real analytic, a potential shortcoming of Ulam's method is that it does not fully exploit this extra structure. To some extent this

[2]where, as is standard, the Perron-Frobenius operator is for an induced expanding Markov map on the unstable boundaries of the Markov partition

[3]This is the Perron-Frobenius operator for T itself, acting on a certain space of generalised functions.

can be remedied by modifying $P_{\mathcal{J}}$ so as to project onto a space of functions which are piecewise linear, or piecewise polynomial of some fixed degree, with respect to $\mathcal{J}$. The resulting modified Ulam scheme has been considered by Li [Li] and Ding & Li [DL1, DL2], and appears to yield a slightly improved rate of convergence to the invariant density.

In the case where T is Markov and piecewise C^ω however, a more natural approach to computing ϱ is the following.

4.3. **Computing metric entropy: The Finite Section method.** Suppose that the expanding Markov map T is piecewise C^ω on its partition $\mathcal{I}$. The finite section method for computing the density function ϱ of T hinges, like the Ulam scheme, on a finite rank approximation to the Perron-Frobenius operator. The difference, however, is that analyticity is exploited in a key way.

We henceforth assume for simplicity that T is a *full branch* expanding map, so that each inverse branch T_i is defined on the whole of I. The real analyticity hypothesis means we can choose a simply connected complex domain $D \supset I$ such that each inverse branch T_i has a holomorphic extension, also denoted T_i, to D, such that $\sup_{z\in D} |T_i'(z)| < 1$ for each $i = 1, \dots, d$. This ensures that $\cup_{i=1}^d T_i(D)$ is compactly contained in D.

In general it may not be possible to choose D as a Euclidean disc, since T_i may have singularities close to I in the complex plane. However we can always reduce to this case by conjugating T with a Riemann mapping from D to the unit disc (see eg. [JP2]). Indeed without loss of generality we shall take D to be the open unit disc centred at 0, and I some closed sub-interval of $(-1, 1)$.

The *Hilbert Bergman space* $A_2(D)$ consists of those holomorphic functions $\varphi : D \to \mathbb{C}$ such that

$$\int_D |\varphi|^2 \, dA < \infty ,$$

where A denotes area measure in the plane (cf. [HKZ]). This is a Hilbert space when equipped with the inner product $(\varphi, \psi)_{A_2(D)} := \int_D \varphi\overline{\psi}\, dA$. The formula (4) defining the Perron-Frobenius operator $\mathcal{L}$ can be extended to the unit disc by holomorphic continuation of the T_i and $|T_i'|$, and $\mathcal{L}$ defines a bounded operator on $A_2(D)$.

Now $A_2(D)$ admits a conveniently explicit orthonormal basis $\{e_i\}_{i=0}^\infty$ of monomials $e_i(z) = z^i$, and for each n we can define a rank-n projection operator

$$\begin{aligned} \pi_n : &A_2(D) \to A_2(D) \\ &\varphi \mapsto \sum_{i=0}^{n-1} (\varphi, e_i) e_i \, . \end{aligned}$$

The n^{th} *finite section* $\mathcal{L}_n$ of the Perron-Frobenius operator $\mathcal{L}$ is then defined by:

$$\mathcal{L}_n := \pi_n \mathcal{L} \pi_n \, .$$

Although the projections π_n only converge *pointwise* to the identity, the finite sections $\mathcal{L}_n$ enjoy the stronger *norm convergence* to $\mathcal{L}$, the norm in question being the induced operator norm on the space $[A_2(D)]$ of bounded linear operators from $A_2(D)$ to itself. Indeed this norm convergence is at an exponential rate (cf. [BJ]):

Proposition 13. *For* $n \geq 1$,

$$||\mathcal{L} - \mathcal{L}_n||_{[A_2(D)]} = O(\alpha^n)$$

for some $0 < \alpha < 1$.

Each $\mathcal{L}_n$ is of rank n, its range being the invariant subspace V_n of polynomials of degree $n-1$. The $n \times n$ matrix M_n representing the operator $L_n := \mathcal{L}_n|_{V_n}$ is precisely the top left $n \times n$ corner of the infinite matrix representing $\mathcal{L}$ with respect to the basis $\{e_i\}_{i=0}^{\infty}$. For n sufficiently large the leading eigenvalue of M_n is close to 1 and is simple, and the corresponding eigenvector ϱ_n is a degree-$(n-1)$ polynomial approximation to the density function ϱ. Classical spectral perturbation theory then implies that $||\varrho - \varrho_n||_{A_2(D)}$ is bounded by a constant multiple of $||\mathcal{L} - \mathcal{L}_n||_{[A_2(D)]}$:

Theorem 14. *If ϱ_n is the leading eigenvector of the n^{th} finite section $\mathcal{L}_n$ of the Perron-Frobenius operator $\mathcal{L}$, then*

$$||\varrho - \varrho_n||_{A_2(D)} = O(\alpha^n) \quad \textit{as } n \to \infty \tag{8}$$

for some $0 < \alpha < 1$.

The constant α is related to the extent to which the holomorphically extended T_i contract the unit disc D inside itself; the stronger the contraction, the smaller α is. The implicit constant in the asymptotic (8) can be estimated by consideration of the resolvents of $\mathcal{L}$ and $\mathcal{L}_n$. In practice it is difficult to obtain sharp bounds on the norms of these resolvents (though see Bandtlow [Ba] for new ideas on this problem), so that explicit error bounds on an approximate density ϱ_n may be rather conservative.

4.4. **Computing metric entropy: Iterating the Perron-Frobenius operator.** Let T be a piecewise C^k expanding Markov map for some $k \geq 2$. By Proposition 9, the value 1 is a simple isolated eigenvalue for the Perron-Frobenius operator $\mathcal{L}$ acting on $C_{\mathcal{I}}^{k-1}(I)$, all other spectral values being of modulus $\leq \alpha$ for some $0 < \alpha < 1$. From the spectral decomposition of $\mathcal{L}$ it follows that for any $g \in C_{\mathcal{I}}^{k-1}(I)$,

$$||\mathcal{L}^n g - \varrho \int g \, d\mu||_{C_{\mathcal{I}}^{k-1}(I)} = O(\alpha^n) \quad \text{as } n \to \infty. \tag{9}$$

Choosing $g = 1$ we see that $\mathcal{L}^n(1) \to \varrho$ in the C^{k-1} norm, the rate of convergence being $O(\alpha^n)$. The iterates $\mathcal{L}^n(1)$ are computable, and for sufficiently large n the Rohlin Lemma implies that

$$\left| \int \log |T'(x)| \mathcal{L}^n(1)(x) \, dx - h(\mu) \right| = O(\alpha^n) \quad \text{as } n \to \infty.$$

A slight variation on this method is to select some arbitrary $y \in I$, then choosing $g = \log |T'|$ and $g = 1$ in (9) we have

$$H_n(y) := \frac{\mathcal{L}^n(\log |T'|)(y)}{\mathcal{L}^n(1)(y)} \longrightarrow \frac{\varrho(y) \int \log |T'| \, d\mu}{\varrho(y) \int 1 \, d\mu} = h(\mu) \quad \text{as } n \to \infty \, ,$$

the rate of convergence again being $O(\alpha^n)$. The quantity $H_n(y)$ can be computed in terms of the n^{th} preimages of y, so that

$$H_n(y) = \frac{\sum_{x \in T^{-n}(y)} \log |T'(x)| / |(T^n)'(x)|}{\sum_{x \in T^{-n}(y)} 1 / |(T^n)'(x)|} \longrightarrow h(\mu) \tag{10}$$

exponentially fast.

A very similar approach uses period-n points instead of n^{th} preimages. Bowen's spatial equidistribution theorem [Bo1], [Bo2] asserts that the sequence of atomic T-invariant probability measures

$$m_n := \frac{\sum_{x \in \text{Fix}(n)} \delta_x / |(T^n)'(x)|}{\sum_{x \in \text{Fix}(n)} 1 / |(T^n)'(x)|}$$

converges to μ in the weak* topology, where $\text{Fix}(n)$ denotes the set of fixed points of T^n. Consequently the sequence

$$h_n := \int \log |T'|\, dm_n = \frac{\sum_{x\in\text{Fix}(n)} \log |T'(x)|/|(T^n)'(x)|}{\sum_{x\in\text{Fix}(n)} 1/|(T^n)'(x)|} \tag{11}$$

converges to $h(\mu) = \int \log |T'|\, d\mu$. Once again the convergence is exponential in n, the approximation (11) being of the same quality as the very similar formula (10).

Although the use of periodic points as opposed to preimages offers no algorithmic advantage in this case, it turns out that a more elaborate regrouping of the periodic points does offer a significantly faster computational scheme, as we shall describe in the next subsection.

4.5. **Computing metric entropy: Periodic orbits.** Here we assume that T is a piecewise C^ω expanding Markov map. If f is real analytic, and D is a suitable complex neighbourhood of I, then the transfer operator $\mathcal{L}_f : A_2(D) \to A_2(D)$ in fact enjoys the following very useful property (cf. [Ru1], [JP2]).

Proposition 15. *The singular values*

$$s_n(\mathcal{L}_f) := \inf \{||\mathcal{L}_f - K|| : K \in [A_2(D)], \text{rank } K < n\}$$

form an exponentially decaying sequence.

In particular $\mathcal{L}_f$ is trace class (i.e., $\sum_n |s_n(\mathcal{L}_f)| < \infty$), and hence compact.

Consequently (see [Sim]) $\mathcal{L}_f$ has a trace $\text{tr}(\mathcal{L}_f) = \sum_n s_n(\mathcal{L}_f)$, and $I - z\mathcal{L}_f$ has a determinant, expressible (cf. [Ru1]) as

$$\det(I - z\mathcal{L}_f) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(\mathcal{L}_f^n) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in\text{Fix}(n)} \frac{\exp\left(f^n(x)\right)}{1 - [(T^n)'(x)]^{-1}}, \tag{12}$$

the trace formula $\text{tr}(\mathcal{L}_f^n) = \sum_{x\in\text{Fix}(n)} \frac{e^{f^n(x)}}{1-[(T^n)'(x)]^{-1}}$ following from the observation that $\mathcal{L}_f^n$ is a sum of weighted composition operators (cf. [May1], [May2]). As before $\text{Fix}(n)$ denotes the fixed point set of T^n. The power series in (6) has a finite radius of convergence, but the determinant itself can be holomorphically continued to an entire function of z. Importantly each $\sum_{x\in\text{Fix}(n)} \frac{\exp(f^n(x))}{1-[(T^n)'(x)]^{-1}}$ is computable provided all period-n points can be located.

Of particular importance are the transfer operators for the family of functions $f_s = s \log |T'|$, and their corresponding determinants

$$\Delta(z, s) = \det(I - z\mathcal{L}_{f_s}) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in\text{Fix}(n)} \frac{|(T^n)'(x)|^s}{1 - [(T^n)'(x)]^{-1}}\,, \tag{13}$$

because the metric entropy $h(\mu)$ can be expressed as:

Proposition 16.

$$h(\mu) = \int \log |T'|\, d\mu = \frac{\partial_s \Delta(1, -1)}{\partial_z \Delta(1, -1)}\,. \tag{14}$$

The proof (see [JP2], Prop. 2.2) of this characterisation of entropy uses thermodynamic formalism, in particular Ruelle's formula (p. 133 of [Ru2]) for the derivative of pressure P together with the fact that e^P is the leading eigenvalue of $\mathcal{L}_f$, and the reciprocal of the smallest zero for Δ.

The computational utility of (14) stems from the fact that Δ is an entire function of z. Indeed the exponential decay of the singular values $s_n(\mathcal{L}_{f_s})$ implies (cf. Lemma 5.2 of [JP2]) that the Taylor coefficients of Δ, and of its partial derivatives $\partial_z \Delta$, $\partial_s \Delta$, are $O(\alpha^{n^2})$ for some $0 < \alpha < 1$. These Taylor coefficients are computable in terms of periodic points of T, by virtue of (13). Therefore (14) can be approximated up to an error $O(\alpha^{n^2})$ by the evaluation at $(z, s) = (1, -1)$ of a degree-n rational function whose coefficients are explicitly computable in terms of the points of period up to n.

Explicitly, we can define signed invariant measures

$$\nu_M = \sum_{\substack{(n_1,\dots,n_m)\\ n_1+\dots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^{m} \sum_{x \in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1\\ j\neq i}}^{m} \sum_{z \in \mathrm{Fix}(n_j)} r(z, n_j) \right) \frac{\delta_x}{|(T^{n_i})'(x) - 1|}$$

where the first summation is over ordered m-tuples of positive integers whose sum is not greater than M, and

$$r(x, n) = \frac{1}{n|(T^n)'(x) - 1|}\,.$$

Each ν_M is an atomic measure, supported on those periodic points of period at most M. The formula for ν_M is the more elaborate regrouping of periodic points mentioned at the end of §4.4. It is derived by first computing the Taylor coefficients for Δ from (13), then differentiating and evaluating at $(1, -1)$. It is noteworthy that the formula for ν_M uses all points of period *up to* M, whereas (11) defining h_M uses only those points of period *exactly* M. Since the number of period-M points is exponential in M, the time required to compute ν_M is of the same order as for h_M.

Normalising the ν_M gives a sequence μ_M of explicitly computable atomic signed invariant *probability* measures, with the property that (see [JP1]):

Theorem 17.

$$\left| h(\mu) - \int \log |T'| \, d\mu_M \right| = O(\alpha^{M^2}) \quad \text{as } M \to \infty$$

for some $0 < \alpha < 1$.

This result represents a significant improvement over the other methods for computing entropy described in this section. For the same order of computational effort, the quality of the entropy approximation is $O(\alpha^{n^2})$ as opposed to $O(\alpha^n)$. Theorems 12, 14 and 17 were phrased so as to facilitate a direct comparison. The computational effort required to compute the Ulam approximant $\mathcal{L}_{\mathcal{J}_n}$ of an n^{th} refined partition, or an n^{th} finite section $\mathcal{L}_n$, or the points of period up to n, is $O(K^n)$ in each case, for some $K > 1$. The periodic point algorithm of this subsection therefore has time complexity $O(\exp(c\sqrt{N}))$ for some $c > 0$, compared to $O(\exp(cN))$ for the other algorithms.

Hyperbolic or expanding dynamical systems are characterised by the fact that they expand or contract distances at exponential speed. It is therefore natural that algorithms for computing invariants which measure exponential growth rates, such as entropy and Lyapunov exponents, should be of exponential complexity. It is rather remarkable on the other hand that *sub-exponential* algorithms such as the periodic point method can be found.

From a philosophical point of view the use of periodic points to compute entropy is satisfyingly canonical, as opposed to the somewhat arbitrary choice of partition in Ulam's method, or of an orthonormal basis in the finite section method. On the other hand these latter two methods have the advantage that the density function ϱ is computed directly. The Ulam method has the additional advantage that the map T need only be piecewise smooth. The finite section method has the advantage of still working well if T is a Markov map with a *countably infinite* Markov partition, such as Gauss's continued fraction map (see for example [DFV]). By contrast both the Ulam method (where the natural partitions are infinite) and periodic point method (where computation of the traces tr$(\mathcal{L}_f^n)$ is problematic due to the infinity of points of given period) fare less well in this context.

5. Variation of entropy and invariant densities

A natural question is to consider the continuity and smoothness properties of the maps $T \mapsto h(\mu_T)$ and $T \mapsto \varrho_T$, defined on appropriate map spaces. In the real analytic case let $EC^\omega_{\mathcal{I}}(I,I)$ denote the space of full branch C^ω expanding maps with partition $\mathcal{I}$, and $C^\omega(I,\mathbb{R})$ the space of real analytic functions on I. We then have (cf. [JP3]).

Theorem 18. *The two maps*

$$\begin{aligned} EC^\omega_{\mathcal{I}}(I,I) &\to C^\omega(I,\mathbb{R}) \\ T &\mapsto \varrho_T \end{aligned}$$

and

$$\begin{aligned} EC^\omega_{\mathcal{I}}(I,I) &\to \mathbb{R} \\ T &\mapsto h(\mu_T) \end{aligned}$$

are both C^ω.

For specific C^ω parametrised families T_ε we can explicitly compute coefficients in the Taylor series for $h(\mu_{T_\varepsilon})$. For example if

$$T_\varepsilon(x) = 2x + \varepsilon \sin 2\pi x \text{ (mod 1)}, \text{ for } -\frac{1}{2\pi} < \varepsilon < \frac{1}{2\pi}$$

then we have the following second order asymptotic (see [JP1]).

Proposition 19. $h(\mu_{T_\varepsilon}) = \log 2 - \frac{\pi^2}{4}\varepsilon^2 + O(\varepsilon^3)$ *as* $\varepsilon \to 0$.

On the other hand we can accurately compute $h(\mu_{T_\varepsilon})$, by the periodic point method of §4.5, say. In Figure 1 these computed values are plotted against the second order approximation obtained above.

In the case where T is piecewise C^k, its density function ϱ_T is C^{k-1}, though we can also choose to view it as lying in any of the spaces C^r, for $0 \le r \le k-1$. We then have (cf. [JP3]):

Theorem 20. *If $1 \le r \le k-1$ then the map*

$$\begin{aligned} &EC^k_{\mathcal{I}}(I,I) \to C^r(I,\mathbb{R}) \\ &T \mapsto \varrho_T \end{aligned}$$

is C^{k-1-r}, while if $r = 0$ then the map is C^{k-2} and its $(k-2)^{th}$ derivative has modulus of continuity $O(x \log x)$.

The entropy map

$$\begin{aligned} &EC^k_{\mathcal{I}}(I,I) \to \mathbb{R} \\ &T \mapsto h(\mu_T) \end{aligned}$$

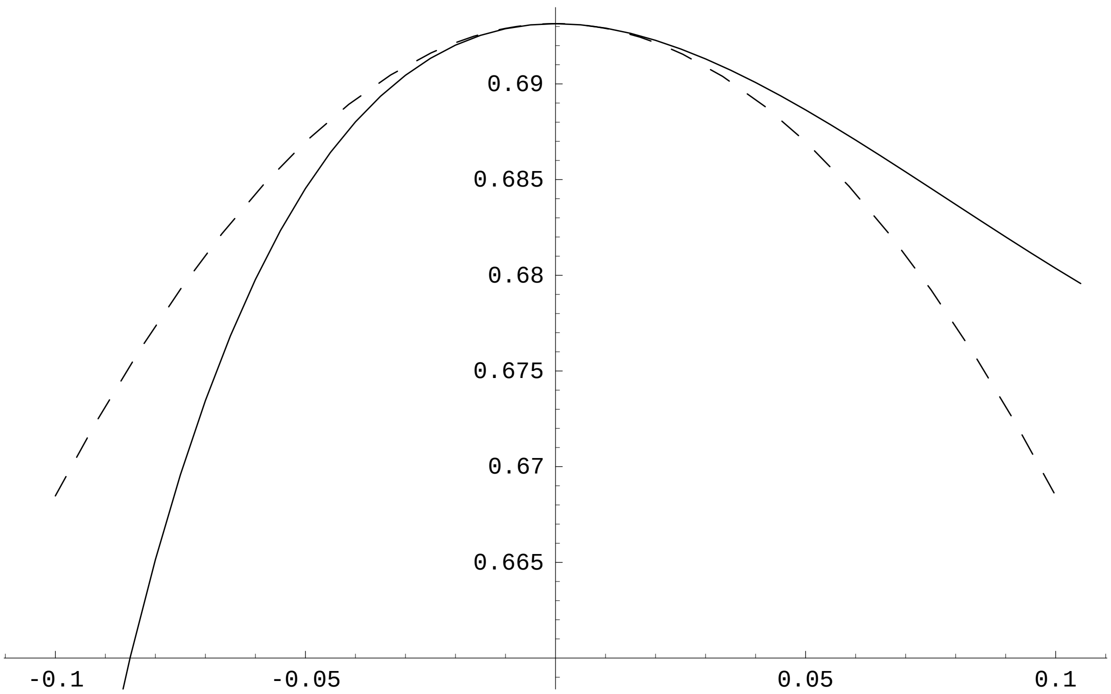

FIGURE 1. Entropy $h(\mu_{T_\varepsilon})$ and second order approximation $\log 2 - \frac{\pi^2}{4}\epsilon^2$ (dashed)

is C^{k-1}.

6. PERIODIC POINTS FOR C^k MAPS

A limitation of the periodic point method described in subsection 4.5 is that the expanding Markov map T must be piecewise C^ω rather than piecewise C^k for finite k. On the other hand, the formula for the atomic signed measures ν_M only involves the *first* derivative of T, so formally makes sense when T is merely piecewise C^k. It is tempting, therefore, to apply the periodic point method verbatim in the case of finite smoothness. However the resulting sequence of approximations to $h(\mu)$ is only exponentially, rather than super-exponentially, convergent. The reason for this is that the determinant $\Delta(z,s) = \det(I - z\mathcal{L}_{f_s})$ defined by (13) is not an entire function of z, so its Taylor coefficients decay only exponentially fast. Indeed strictly speaking $\Delta(z,s)$ is not a determinant in the C^k case, since the transfer operator $\mathcal{L}_{f_s}$ is not trace class (nor even compact, though it is quasi-compact, cf. Proposition 10).

If T is merely piecewise C^k then an alternative method for computing its entropy is to first approximate T by a sequence of C^ω expanding maps T_m. Such an approximation is possible in various topologies, in particular the C^k topology. For example if $T : I \to I$ is the lift of a C^k circle expanding map (so in particular T is full branch) then this approximation is explicitly achieved by convolving T with suitable summability kernels, for example the Féjer kernel

$$K_m(x) = \frac{1}{m+1}\left(\frac{\sin{(n+1)x/2)}}{\sin{(x/2)}}\right)^2 .$$

If $T(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ then the resulting convolution

$$T_m(x) = \frac{1}{m+1}\sum_{l=0}^{m}\sum_{j=-l}^{l} a_j e^{2\pi i j x}$$

is a trigonometric polynomial, and in particular real analytic. Moreover,

$$D^k\left(T - T_m\right)(x) = (D^k T)(x) - (D^k T) * K_m(x) ,$$

and since $D^k T$ is a continuous function then $||D^k T - D^k(T * K_m)||_\infty \to 0$, as $m \to +\infty$, by classical properties of the Féjer kernel (cf. p. 18 of [Katz]).

Having located a C^ω map S which C^k-approximates the C^k map T, the entropy $h(\mu_S)$, which is computable using periodic points, is close to the desired entropy $h(\mu_T)$ by Theorem 20. Similarly the density ϱ_S is close to ϱ_T, and if ϱ_S can be computed via periodic points then this provides an approximation to ϱ_T.

We now describe how to build up the invariant density of a C^ω expanding Markov map from its periodic points. Suppose for simplicity that T is full branch. The density function ϱ is real analytic, but we can view it as an element of $L^2(I, dx)$, and expand $\varrho = \sum_n (\varrho, f_n) f_n$ in terms of a convenient orthonormal basis $\{f_n\}$.

The most natural basis consists of orthogonal polynomials. If we parameterize the interval as $I = [-1, 1]$ then these are the Legendre polynomials

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \dots\right\},$$

which we normalise by setting $f_n = (n + \frac{1}{2})^{1/2} P_n$. Similarly, if T were a map of the circle then it would be natural to expand ϱ as a Fourier series.

Each expansion coefficient $(\varrho, f_n) = \int \varrho(x) f_n(x)\, dx$ can be computed with error $O(\alpha^{M^2})$ from the points of period up to M, by close analogy with the method outlined in §4.5 (f_n rather than $\log|T'|$ is integrated with respect to the atomic measures μ_M, see [JP2]). Combining the first few such approximate coefficients $(\varrho, f_n)^{(M)}$ yields an approximation $\varrho_N^{(M)} := \sum_{n=0}^{N} (\varrho, f_n)^{(M)} f_n$ to ϱ. The following result (see [JP2]) gives bounds on the quality of this approximation:

Theorem 21. *If $M := M(N) = O(\exp(CN^2))$ for some $C > 0 =$ then*

$$||\varrho - \varrho_N^{(M)}||_{C^0} = O(\alpha^N)$$

for some $0 < \alpha < 1$.

7. Calculation of entropies for Anosov flows

In this final section, we shall complete the circle of ideas, returning to the case of the entropy of geodesic flows on manifolds of negative curvature. The periodic point method used to compute the metric entropy for expanding maps in §4.5 can be adapted to such geodesic flows, or to more general Anosov flows.

Assume that $\phi_t : M \to M$ is a three dimensional C^ω Anosov flow with SRB measure μ. We would like to estimate both the topological and metric entropy using periodic orbit data. Other approaches to estimating entropies can be found in work of Freire and Mañé [FM], Kozlovski [Koz], Osserman and Sarnak [OS].

Let us make the simplifying assumption that the horocycle foliations for the flow are C^ω, so that we can take C^ω Markov transverse sections $\mathcal{T} = \{T_1, \ldots, T_k\}$. We may further assume that each section is foliated by pieces of stable manifold. This has the advantage that the sections quotient to intervals, and the Poincaré map between sections quotients to a C^ω map T on the disjoint union of the intervals. The return time function gives rise to a C^ω function r on the union of these intervals. The topological entropy h of the Anosov flow is then characterized by $P(-hr) = 0$.

A closed orbit τ for the flow corresponds to a periodic orbit $\{x, Tx, \ldots, T^{n-1}x\}$. Let us write $|\tau| = n$. In the case of geodesic flows, if one uses the Bowen-Series coding then it can be arranged so that $|\tau|$ has the topological interpretation as the word length of the associated curve in the fundamental group. We can write $\lambda(\tau) = r^n(x)$. The expansion in the unstable direction around the orbit is given by $\lambda^u(x) = \log |(T^n)'(x)|$. We associate a zeta function

$$\zeta(z,s) = \prod_\tau \left(1 - z^{|\tau|} e^{-s\lambda(\tau)}\right)^{-1} ,$$

and the determinant of the associated transfer operator is

$$\begin{aligned} d(z,s) &:= \exp - \sum_{n=1}^\infty \frac{z^n}{n} \sum_{T^n x = x} \frac{e^{-sr^n(x)}}{1 - |(T^n)'(x)|^{-1}} \\ &= \exp - \sum_{n=1}^\infty z^n \sum_{|\tau|=n} \frac{e^{-s\lambda(\tau)}}{(1 - e^{-\lambda^u(\tau)})} \\ &= 1 + \sum_{n=1}^\infty a_n(s) z^n , \end{aligned}$$

defined initially for $|z| < e^{P(-Re(s)r)}$ but admitting an analytic extension to the entire z-plane. Moreover, the Taylor coefficients $a_n(s)$ can be written

$$a_n(s) = \sum_{|\lambda_1| + \cdots + |\lambda_k| = n} (-1)^k e^{-s(\lambda(\tau_1) + \cdots + \lambda(\tau_k))} ,$$

with the estimate $a_n(s) = O(\theta^{n^2})$ for $0 < \theta < 1$. From §3.3 we know that $s = h(\phi)$ is a pole of $\zeta(1,s)$, from which we deduce that it is also a zero for $d(1,s)$.

Proposition 22. *Solutions s_N to $\sum_{n=1}^N a_n(s) = 1$ satisfy $|s_N - h(\phi)| = O(\theta^{N^2})$ as $N \to +\infty$, for some $0 < \theta < 1$.*

Similarly we can apply the methods of subsection 4.5 to the interval map $T : I \to I$ to get a new expression for the metric entropy of the SRB measure μ, namely

$$\int \log |T'| d\mu$$
$$= \lim_{M \to +\infty} \frac{\sum_{\substack{(n_1,\dots,n_m) \\ n_1+\dots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{|\tau|=n_i} \left(\prod_{\substack{j=1 \\ j\neq i}}^m \sum_{|\tau'|=n_j} r(\tau', n_j) \right) \frac{\lambda^u(\tau)}{|\lambda^u(\tau)-1|}}{\sum_{\substack{(n_1,\dots,n_m) \\ n_1+\dots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^m \sum_{|\tau|=n_i} \left(\prod_{\substack{j=1 \\ j\neq i}}^m \sum_{|\tau'|=n_j} r(\tau', n_j) \right) \frac{1}{|\lambda^u(\tau)-1|}}$$

where
$$r(\tau, n) = \frac{1}{n|\lambda^u(\tau) - 1|}.$$

For general C^ω Anosov flows, however, the foliations are typically not C^ω. Nevertheless, in this case we can turn to the analysis of Rugh [Rug1], [Rug2], who associated to the general class of flows and their Markov sections a slightly different transfer operator, which is still nuclear. In this case the weight associated to each closed orbit $\{x, Tx, \dots, T^{n-1}x\}$ or τ is $e^{-s\lambda(\tau)}/(1 - \lambda^u(\tau)^{-1})(1 - \lambda^s(\tau)^{-1})$, where $\lambda^s(\tau) > 1$ is the expansion in the stable direction around τ. For example the determinant of the associated transfer operator can be written

$$d^u(z, s) := \exp - \sum_{n=1}^{\infty} z^n \sum_{|\tau|=n} \frac{e^{-s\lambda(\tau)}}{(1 - e^{-\lambda^u(\tau)})(1 - e^{-\lambda^s(\tau)})}$$
$$= 1 + \sum_{n=1}^{\infty} b_n(s) z^n,$$

where again $b_n(s) = O(\theta^{n^2})$ for some $0 < \theta < 1$. We know that $s = h(\phi)$ is a solution to $\zeta(1, s)^{-1} = 0$ and as before it is also a zero for $d^u(1, s)$. We therefore have

Proposition 23. *Solutions s_N to $\sum_{n=1}^N b_n(s) = 1$ satisfy $|s_N - h(\phi)| = O(\theta^{N^2})$ as $N \to +\infty$, for some $0 < \theta < 1$.*

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