ON THE DENSITY OF HAUSDORFF DIMENSIONS OF BOUNDED TYPE CONTINUED FRACTION SETS: THE TEXAN CONJECTURE

Oliver Jenkinson

Queen Mary, University of London

Abstract. Given a non-empty finite subset A of the natural numbers, let E_A denote the set of irrationals $x \in [0,1]$ whose continued fraction digits lie in A. In general E_A is a Cantor set whose Hausdorff dimension $\dim(E_A)$ is between 0 and 1. It is shown that the set $\mathcal{D} = \{\dim(E_A) : A \subset \mathbb{N} \text{ is finite}\} \subset [0,1]$ intersects $[0, \frac{1}{2}]$ densely.

We then describe a method for accurately computing dimensions $\dim(E_A)$, and employ it to investigate numerically the way in which $\mathcal{D}$ intersects $[\frac{1}{2}, 1]$. These computations tend to support the conjecture, first formulated independently by Hensley, and by Mauldin & Urbański, that $\mathcal{D}$ is dense in $[0,1]$.

In the important special case $A = \{1,2\}$, we use our computational method to give an accurate approximation of $\dim(E_{\{1,2\}})$, improving on the one given in [JP1].

1. Introduction

Given a non-empty finite subset A of the natural numbers, let E_A denote the set of irrationals $x \in [0,1]$ whose continued fraction partial quotients all lie in A. That is, E_A consists of those numbers

$$x = [0; a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where each $a_i \in A$. A set of the form E_A is said to be of *bounded type*. An entertaining and wide-ranging survey of continued fractions with bounded partial quotients is given by Shallit [Sha].

If A is a singleton then so is E_A. Otherwise, E_A is a Cantor set whose Hausdorff dimension $\dim(E_A)$ lies strictly between the values 0 and 1. Interest in the Hausdorff dimension of bounded type continued fraction sets was motivated historically by connections with Diophantine approximation, in particular with the Markoff and Lagrange spectra (see [Bu1, Bu2, CuF]).

Let

$$\mathcal{D} := \{\dim(E_A) : A \text{ is finite}\}$$

be the (countable) set of all possible Hausdorff dimensions of bounded type continued fraction sets. Clearly $0 \in \mathcal{D}$, corresponding to the case where A is a singleton, but no other members of $\mathcal{D}$ are known explicitly. It is natural to enquire about the *accumulation points* of the set $\mathcal{D}$. We prove:

Theorem 1. *$\mathcal{D}$ intersects $[0, \frac{1}{2}]$ densely.*

This result was the focus of the unpublished preprint [J]. Shortly after writing it, Dan Mauldin drew our attention to his work with Mariusz Urbański on infinite iterated function systems, in particular to their proof of Theorem 1 which appears as Theorem 6.3 in [MU2]. While acknowledging this precedence, we retain our original proof of Theorem 1 in this updated version of [J], it being hopefully more leisurely and self-contained than that of [MU2]. In particular we do not explicitly use the technology or terminology of infinite conformal iterated function systems developed in [MU1, MU2].

Our own interest in the set $\mathcal{D}$ stems from the 1996 paper of Hensley [Hen2], in which various conjectures about the Hausdorff dimension of bounded type continued fraction sets were formulated. In particular his Conjecture 2 [Hen2, p. 15] was also arrived at independently 3 years later, some 200 miles north through the prairies and lakes of Texas, by Mauldin & Urbański (Problem 3 in [MU2, p. 5024]):

The Texan Conjecture. *$\mathcal{D}$ is a dense subset of* $[0, 1]$.

We go on to consider whether or not there is any evidence in favour of the Texan conjecture, by computing the Hausdorff dimension of a large number of sets E_A. Although it is not feasible to compute any $\dim(E_A)$ *exactly* (unless A is a singleton), it is possible to make high quality computer-assisted approximations. In §4 we describe a mathematical algorithm, coming from joint work with Mark Pollicott [JP1, JP2], which allows us to do just that. The resulting computations are presented in §5, and do lend considerable support to the Texan conjecture.

2. Continued fractions and thermodynamic formalism

The Hausdorff dimension $\dim(E)$ of any set $E \subset \mathbb{R}$ is defined as follows (see [Fal]). For any $\delta > 0$, let $H^\delta_\varepsilon(E) = \inf_{\mathcal{U}}\{\sum_i \operatorname{diam}(U_i)^\delta\}$, where the infimum is taken over all countable covers $\mathcal{U} = \{U_i\}$ of E satisfying $\operatorname{diam}(U_i) \le \varepsilon$ for each i. Set $H^\delta(E) = \lim_{\varepsilon\to 0} H^\delta_\varepsilon(E)$, and then define

$$\dim(E) = \inf\{\delta : H^\delta(E) = 0\}.$$

It is known that the Hausdorff dimension of sets of the form E_A can be studied using the thermodynamic formalism of Ruelle [Ru2]. Define $T_j(x) = 1/(j + x)$, for $j \in \mathbb{N}$. Let A be any subset of $\mathbb{N}$. For $\underline{i} = (i_1, \dots , i_n) \in A^n$, let $T_{\underline{i}}$ denote the composition $T_{i_1} \circ \dots \circ T_{i_n}$. Then we can write $E_A = \bigcap_{n=1}^\infty \bigcup_{\underline{i}\in A^n} T_{\underline{i}}(I)$, and this set is invariant under the Gauss map $T(x) = 1/x \pmod 1$.

Let $x_j \in [0, 1]$ denote the unique fixed point of the contraction T_j. Its continued fraction expansion is $x_j = [0; j, j, j, \dots]$. Of importance will be the quantities $|T_j'(x_j)|$, for which we have the following inequality:

$$1/(j^2 + j + 1) < |T_j'(x_j)| = \frac{2}{2 + j^2 + j\sqrt{j^2 + 4}} < 1/(j^2 + 1).$$

Clearly $|T_j'(x_j)| \to 0$ as $j \to \infty$, and

$$\sum_{j=1}^{\infty} |T_j'(x_j)|^s = \infty \quad \text{if and only if} \quad s \le 1/2. \tag{2.1}$$

Let $x_{\underline{i}}$ denote the unique fixed point of the composition $T_{\underline{i}}$. Its continued fraction expansion $[0; i_1, \ldots, i_n, i_1, \ldots, i_n, \ldots]$ is periodic, so that $x_{\underline{i}}$ is a quadratic irrational. For $s \in [0,1]$, we introduce the *pressure* (see [Ru, PP])

$$P_A(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\underline{i}\in A^n} |T'_{\underline{i}}(x_{\underline{i}})|^s \right).$$

Clearly $P_A(s)$ is a non-increasing function of s, since each $|T'_{\underline{i}}(x_{\underline{i}})| < 1$. If A is infinite then $P_A(s) = \infty$ for sufficiently small s, while if $s > 1/2$ then $P_A(s) < \infty$ for all A. When restricted to $\{s \in [0,1] : P_A(s) < \infty\}$, each map $s \mapsto P_A(s)$ is real-analytic, strictly decreasing, and strictly convex (see [MU1], Prop 3.3).

For $s \geq 0$, define the *transfer operator* (see [Ru, PP]) $\mathcal{L}_{A,s}$, acting on the space of continuous complex-valued functions of $[0,1]$, by

$$\mathcal{L}_{A,s} w(z) = \sum_{j\in A} |T'_j(z)|^s w(T_j z) = \sum_{j\in A} \frac{w(T_j z)}{(j+z)^{2s}},$$

whenever s is large enough to ensure convergence of the sum (in the case where A is infinite).

The key result relating the Hausdorff dimension $\dim(E_A)$ to the operator $\mathcal{L}_{A,s}$ and the pressure $P_A(s)$ is:

Lemma 1. *Let A be a finite non-empty subset of $\mathbb{N}$.*

(a) $\mathcal{L}_{A,s}$ has a simple eigenvalue $\lambda_A(s)$ of maximum modulus, and $\lambda_A(s) = \exp(P_A(s)) = \lim_{n\to\infty} \left(\mathcal{L}^n_{A,s}[1](0)\right)^{1/n}$.

(b) $\dim(E_A)$ is the unique solution s to the equation $\lambda_A(s) = 1$; equivalently $\dim(E_A)$ is the unique zero s of $P_A(s)$.

Proof. Part (a) is a classical result of Ruelle (see [Ru2], [PP, Thm 2.2]). Part (b) is due to Bowen (see [B, Fal]).

In the case where $A = \{j\}$ is a singleton, it is easily seen that $P_{\{j\}}(s) = \log |T'_j(x_j)|^s$ and $\lambda_{\{j\}}(s) = |T'_j(x_j)|^s$ (the corresponding eigenspace consists of constant functions).

In fact the transfer operator $\mathcal{L}_{A,s}$ preserves various Banach spaces which are smaller than $C^0(I)$. In particular it is a *compact* operator when restricted to certain spaces of functions which are analytic in some fixed complex neighbourhood of $[0,1]$. There is considerable flexibility in selecting such spaces (see the comments in [JGU, pp. 759–760]). For definiteness let H be the Hilbert space of functions f which are analytic on the disc $D \subset \mathbb{C}$ of radius $3/2$ centred at 1 (cf. [May, JP1, JGU]), and satisfy $||f||^2_H := \int_D |f|^2\, dA < \infty$, where A is two-dimensional Lebesgue measure. In particular this is a *Bergman* space (see [HKZ] for general facts about Bergman spaces, and [BJ], [JP3] for transfer operators acting on Hilbert Bergman spaces). On H, the leading eigenvalue of $\mathcal{L}_{A,s}$ is still $\lambda_A(s) = \exp(P_A(s))$, but now it is an *isolated* point in the spectrum since $\mathcal{L}_{A,s}$ is compact. In fact $\mathcal{L}_{A,s}$ is trace-class (cf. [JGU] for related results in the context of continued fractions, and [BJ], [JP3] for trace-classness on Hilbert-Bergman spaces), a fact we shall require in §4 in order to consider the *determinant* of $\mathcal{L}_{A,s}$.

The importance of the following result is that the upper and lower bounds are independent of the digit set A.

Lemma 2. *For all finite non-empty subsets $A \subset \mathbb{N}$, $s \in [0,1]$, and $n \geq 0$ we have*

$$\frac{1}{4}\lambda_A(s)^n \leq \mathcal{L}_{A,s}^n[1](x) \leq 4\lambda_A(s)^n \tag{2.2}$$

for all $x \in I$.

Proof. Now $T_{\underline{i}}'(z) = (-1)^n/(q_{n-1}z+q_n)^2$, where $p_n/q_n = [0; i_1, \ldots, i_n]$, so that

$$\frac{T_{\underline{i}}'(z)}{T_{\underline{i}}'(x)} = \frac{(q_{n-1}x+q_n)^2}{(q_{n-1}z+q_n)^2} \leq \left(\frac{q_{n-1}}{q_n}+1\right)^2 \leq 4.$$

For any continuous function w, the n^{th} iterate $\mathcal{L}_{A,s}^n w$ is given by the formula

$$\mathcal{L}_{A,s}^n w(z) = \sum_{\underline{i} \in A^n} |T_{\underline{i}}'(z)|^s w(T_{\underline{i}} z).$$

Choosing w to be the constant function $w=1$ gives, for any $x, z \in [0,1]$,

$$\mathcal{L}_{A,s}^n[1](z) = \sum_{\underline{i}\in A^n} |T_{\underline{i}}'(z)|^s \leq \sum_{\underline{i}\in A^n} \left|\frac{T_{\underline{i}}'(z)}{T_{\underline{i}}'(x)}\right|^s |T_{\underline{i}}'(x)|^s \leq \sum_{\underline{i}\in A^n} 4^s |T_{\underline{i}}'(x)|^s = 4^s \mathcal{L}_{A,s}^n[1](x),$$

and since $s \leq 1$ we deduce that

$$\lambda_A(s)^{-n}\mathcal{L}_{A,s}^n[1](z) \leq 4\lambda_A(s)^{-n}\mathcal{L}_{A,s}^n[1](x). \tag{2.3}$$

The image of a strictly positive function under $\mathcal{L}_{A,s}$ is again strictly positive, so that each function $\lambda_A(s)^{-n}\mathcal{L}_{A,s}^n[1]$ is strictly positive. Moreover, by Theorem 2.2(iii) of [PP] we know there is a probability measure m_A, concentrated on E_A, such that

$$\int_{E_A} \lambda_A(s)^{-n}\mathcal{L}_{A,s}^n[1]\, dm_A = \int_{E_A} 1\, dm_A = 1.$$

These two properties of the function $\lambda_A(s)^{-n}\mathcal{L}_{A,s}^n(1)$, together with (2.3), imply that

$$1/4 \leq \lambda_A(s)^{-n}\mathcal{L}_{A,s}^n[1](x) \leq 4$$

for all $x \in [0,1]$ and for all $n \geq 0$, so the lemma is proved.

Our strategy for proving Theorem 1, in spirit the same as that of Mauldin & Urbański [MU2], relies on controlling the increase in dimension which occurs when we add a new digit j to a digit set A. Certainly $\dim(E_A) \geq \dim(E_{A\cup\{j\}})$, and in fact we show that this inequality is strict. Although the increase in dimension depends on both j and A, importantly it can be bounded, above and below, solely in terms of j. The key to this bound is the following proposition, which examines the relation between the maximal eigenvalues $\lambda_A(s)$ and $\lambda_{A\cup\{j\}}(s)$.

Proposition 1. *Fix $s \in [0,1]$. For any finite non-empty digit set $A \subset \mathbb{N}$, and $j \notin A$, we have*

$$\frac{1}{16}|T_j'(x_j)|^s \leq \lambda_{A\cup\{j\}}(s) - \lambda_A(s) \leq 16|T_j'(x_j)|^s.$$

Proof. Let us denote $\mathcal{L}_0 = \mathcal{L}_{A,s}$ and $\mathcal{L}_1 = \mathcal{L}_{\{j\},s}$. We write $\lambda_0 = \lambda_A(s)$ and $\lambda_1 = |T_j'(x_j)|^s$ for the spectral radii of the operators $\mathcal{L}_0$ and $\mathcal{L}_1$.

For $\underline{k} = (k_1, \ldots, k_n) \in \{0,1\}^n$, let $\mathcal{L}_{\underline{k}}$ denote the composition $\mathcal{L}_{k_1} \circ \ldots \circ \mathcal{L}_{k_n}$. Then

$$\mathcal{L}^n_{A\cup\{j\},s} = (\mathcal{L}_0 + \mathcal{L}_1)^n = \sum_{\underline{k}\in\{0,1\}^n} \mathcal{L}_{\underline{k}}. \tag{2.4}$$

Now there are $\binom{n}{r}$ words $\underline{k} \in \{0,1\}^n$ for which the symbol 1 occurs precisely r times, and for such a $\underline{k}$ we can write

$$\mathcal{L}_{\underline{k}} = \mathcal{L}_0^{q_1} \circ \mathcal{L}_1 \circ \mathcal{L}_0^{q_2} \circ \mathcal{L}_1 \circ \ldots \circ \mathcal{L}_0^{q_r} \circ \mathcal{L}_1 \circ \mathcal{L}_0^{q_{r+1}} \tag{2.5}$$

for some powers $q_i \geq 0$ with $q_1 + \ldots + q_{r+1} = n - r$.

By Lemma 2, for all $q \geq 0$ and all x we have

$$\frac{1}{4}\lambda_0^q \leq \mathcal{L}_0^q[1](x) \leq 4\lambda_0^q \quad \text{and} \quad \frac{1}{4}\lambda_1 \leq \mathcal{L}_1[1](x) \leq 4\lambda_1.$$

So from (2.5) we deduce that if the symbol 1 occurs r times in $\underline{k}$ then

$$4^{-(2r+1)}\lambda_0^{n-r}\lambda_1^r \leq \mathcal{L}_{\underline{k}}[1](0) \leq 4^{2r+1}\lambda_0^{n-r}\lambda_1^r.$$

Substituting into (2.4) gives

$$(\mathcal{L}_0 + \mathcal{L}_1)^n\,[1](0) \geq \sum_{r=0}^{n} \binom{n}{r} 4^{-(2r+1)}\lambda_0^{n-r}\lambda_1^r = \frac{1}{4}\left(\lambda_0 + \frac{1}{16}\lambda_1\right)^n, \tag{2.6}$$

and

$$(\mathcal{L}_0 + \mathcal{L}_1)^n\,[1](0) \leq \sum_{r=0}^{n} \binom{n}{r} 4^{2r+1}\lambda_0^{n-r}\lambda_1^r = 4\left(\lambda_0 + 16\lambda_1\right)^n. \tag{2.7}$$

The leading eigenvalue of $\mathcal{L}_{A\cup\{j\},s} = \mathcal{L}_0 + \mathcal{L}_1$ (which is also its spectral radius) is equal to $\lim_{n\to\infty} \left((\mathcal{L}_0 + \mathcal{L}_1)^n\,[1](0)\right)^{1/n}$. Taking n^{th} roots in (2.6) and (2.7), and letting $n \to \infty$, we obtain

$$\lambda_A(s) + \frac{1}{16}|T_j'(x_j)|^s \leq \lambda_{A\cup\{j\}}(s) \leq \lambda_A(s) + 16|T_j'(x_j)|^s,$$

from which the result follows.

3. Proof of Theorem 1

Proposition 1 immediately yields the following two results.

Corollary 1. *Let A be a non-empty finite digit set, not containing the digit j. The Hausdorff dimension of $E_{A\cup\{j\}}$ is strictly larger than the Hausdorff dimension of E_A.*

Proof. Denote $d_A = \dim(E_A)$, so that $\lambda_A(d_A) = 1$. Proposition 1 implies that $\lambda_{A\cup\{j\}}(d_A) > \lambda_A(d_A) = 1$. Since $\lambda_{A\cup\{j\}}(s)$ is decreasing in s, Lemma 1(b) gives $\dim(E_{A\cup\{j\}}) > d_A$.

Corollary 2. *If $A, B \subset \mathbb{N}$ are two finite digit sets, neither of which is a subset of the other, then*

$$dim(E_{A\cup B}) > \max\{dim(E_A), dim(E_B)\}\,.$$

For $j \notin A$, the differences $\dim(E_{A\cup\{j\}}) - \dim(E_A)$ are all strictly positive, by Corollary 1. These differences depend on both A and j, and if we knew them all *exactly* then this would lead to a solution of the Texan conjecture, in its full generality. Similarly, the strictly positive differences $\lambda_{A\cup\{j\}}(s) - \lambda_A(s)$ also depend on both A and j, but again their exact values are inaccessible. The importance of Proposition 1 is that the estimates $\lambda_{A\cup\{j\}}(s) - \lambda_A(s) \in \left[\frac{1}{16}|T_j'(x_j)|^s, 16|T_j'(x_j)|^s\right]$ are *independent of A*. This allows us to exploit the divergence of the sum (2.1) for $s \in [0, \frac{1}{2}]$ in order to prove Theorem 1. First we state an elementary lemma without proof.

Lemma 3. *Suppose $\gamma > 0$. Let $(t_k)_{k\ge1}$ be a sequence of real numbers with $0 < t_k < \gamma$ for each k, and such that $\sum_{k=1}^\infty t_k = \infty$. Then for any $L > \gamma$, there exists $N \ge 1$ such that $\sum_{k=1}^N t_k \in [L-\gamma, L]$.*

Proof of Theorem 1.

Let $d \in [0, \frac{1}{2}]$. We will show that there exists a sequence $A^{(n)}$ of finite digit sets satisfying $\dim(E_{A^{(n)}}) \nearrow d$.

Let (γ_n) be a sequence of positive reals such that $\gamma_n \searrow 0$. The first step is to prove that for each $n \ge 1$, there exists a finite set $A^{(n)}$ such that

$$\lambda_{A^{(n)}}(d) \in [1-\gamma_n, 1]. \tag{3.1}$$

Let $A^{(0)} = \{1\}$.

For each $n \ge 1$, we define $A_1^{(n)}$ by $A_1^{(n)} := A^{(n-1)}$, where the $A^{(n-1)}$ are defined by the following procedure.

If $i \ge 1$ is such that $A_i^{(n)}$ satisfies $\lambda_{A_i^{(n)}}(d) \in [1-\gamma_n, 1]$, then we define $A^{(n)} := A_i^{(n)}$, and (3.1) is satisfied.

If on the other hand $i \ge 1$ is such that $A_i^{(n)}$ satisfies $\lambda_{A_i^{(n)}}(d) < 1-\gamma_n$ then we set

$$j_i^{(n)} := \min\left\{ j > \max A_i^{(n)} : \lambda_{A_i^{(n)}\cup\{j\}}(d) \le 1 \right\},$$

and then define $A_{i+1}^{(n)} = A_i^{(n)} \cup \{j_i^{(n)}\}$. Note that $j_i^{(n)}$ certainly exists, by Proposition 1: adding any symbol $j \notin A_i^{(n)}$ to the alphabet $A_i^{(n)}$ increases the spectral radius of the corresponding transfer operator by at most $16|T_j'(x_j)|^d$, and $|T_j'(x_j)|^d \to 0$ as $j \to \infty$.

We will now show that for each $n \geq 1$ there does exist some $I = I(n) \geq 1$ with $A_I^{(n)}$ satisfying $\lambda_{A_I^{(n)}}(d) \in [1-\gamma_n, 1]$. Since $|T_j'(x_j)|^d \to 0$ as $j \to \infty$, we can choose $J \in \mathbb{N}$ such that $16|T_j'(x_j)|^d < \gamma$ for all $j \geq J$. So if $i \geq 1$ is such that both $\max A_i^{(n)} \geq J$ and $\lambda_{A_i^{(n)}}(d) < 1-\gamma_n$, then Proposition 1 implies that $j_i^{(n)} = 1 + \max A_i^{(n)}$. Therefore

$$A_{i+k}^{(n)} = A_i^{(n)} \cup \{1 + \max A_i^{(n)}, \dots, k + \max A_i^{(n)}\}$$

for any $k \geq 1$ such that $\lambda_{A_{i+k-1}^{(n)}}(d) < 1 - \gamma_n$.

For $k \geq 1$, define $t_k = \lambda_{A_{i+k}^{(n)}}(d) - \lambda_{A_{i+k-1}^{(n)}}(d)$. Then $t_k < 16|T_k'(x_k)|^d < \gamma_n$, by Proposition 1. But Proposition 1 also gives that $t_k > \frac{1}{16}|T_k'(x_k)|^d$, and since $d \in [0, 1/2]$ then by (2.1) we have $\sum_{k=1}^\infty t_k = \infty$.

Applying Lemma 3, with the choices $L = 1 - \lambda_{A_i^{(n)}}(d)$ and $\gamma = \gamma_n$, we deduce there exists $N \geq 1$ such that

$$\lambda_{A_{i+N}^{(n)}}(d) - \lambda_{A_i^{(n)}}(d) = \sum_{k=1}^N t_k \ \in [1 - \lambda_{A_i^{(n)}}(d) - \gamma_n, 1 - \lambda_{A_i^{(n)}}(d)].$$

That is, $\lambda_{A_{i+N}^{(n)}}(d) \in [1-\gamma_n, 1]$. So setting $I(n) := i + N$ gives $\lambda_{A_I^{(n)}}(d) \in [1-\gamma_n, 1]$, as claimed (and we then define $A^{(n)} := A_I^{(n)}$ as described above, so that (3.1) holds).

We have verified that for each $n \geq 1$ there exists a finite digit set $A^{(n)}$ such that (3.1) is satisfied, and hence $\lambda_{A^{(n)}}(d) \nearrow 1$ as $n \to \infty$. Moreover, by construction the sets $A^{(n)}$ form an increasing sequence.

If $\dim(E_{A^{(m)}}) = d$ for some m, then clearly $A^{(n)} = A^{(m)}$ for all $n \geq m$, and we are done. If no such m exists, then $B := \cup_{n=1}^\infty A^{(n)}$ is an infinite digit set.

If H is the Hilbert Bergman space defined in §2 then the operators $\mathcal{L}_{A^{(n)},s} : H \to H$ clearly converge to $\mathcal{L}_{B,s} : H \to H$ in operator norm, uniformly in s on any compact complex neighbourhood of d. The leading eigenvalue of each of these operators is simple, and also isolated (since the operators are compact). Standard spectral perturbation theory (cf. [ALL, Cor. 2.13]) then implies that the function $\lambda_B(\cdot)$ is the uniform limit of $\lambda_{A^{(n)}}(\cdot)$ on a compact neighbourhood of d. Therefore $P_B(\cdot) = \log \lambda_B(\cdot)$ is the uniform limit of $P_{A^{(n)}}(\cdot) = \log \lambda_{A^{(n)}}(\cdot)$ on a compact neighbourhood of d. So $P_B(d) = \lim_{n\to\infty} P_{A^{(n)}}(d) = \lim_{n\to\infty} \log \lambda_{A^{(n)}}(d) = 0$ (which in particular is finite). We claim this implies that

$$\dim(E_{A^{(n)}}) \nearrow d. \tag{3.2}$$

On $[0, 1]$, each $P_{A^{(n)}}(\cdot)$ is strictly decreasing and strictly convex, so $P'_{A(n)}(s) < 0$ for all $s \in [0, 1]$ and all n. Now suppose there exists $\varepsilon > 0$ such that $P'_{A^{(n)}}(d) \leq -\varepsilon$ for all n. The convexity of $s \mapsto P_{A^{(n)}}(s)$ means that its unique zero $\dim(E_{A^{(n)}})$ satisfies

$$\dim(E_{A^{(n)}}) \in \left[d - \frac{P_{A^{(n)}}(d)}{P'_{A^{(n)}}(d)}, d\right] \subseteq \left[d + \frac{P_{A^{(n)}}(d)}{\varepsilon}, d\right],$$

and since $P_{A^{(n)}}(d) \nearrow 0$ we deduce that (3.2) holds.

So the only way (3.2) could fail is if the derivative $P'_{A^{(n)}}(d) \nearrow 0$ as $n \to \infty$. But $P'_B(\cdot)$ is the uniform limit of $P'_{A^{(n)}}(\cdot)$ on a compact neighbourhood of d, so we obtain $P'_B(d) = \lim_{n\to\infty} P'_{A^{(n)}}(d) = 0$. This contradicts the fact that $s \mapsto P_B(s)$ is strictly decreasing and strictly convex, so (3.2) cannot fail, and the theorem is proved.

Remark. The restriction to the set $[0, \frac{1}{2}]$ in Theorem 1, rather than the full interval $[0,1]$, is clearly an artefact of the above method of proof, which hinges on the divergence of the series (2.1).

4. How to compute the dimension of E_A?

In general the Hausdorff dimension of a given bounded type continued fraction set E_A cannot be computed explicitly, though various methods have been devised to provide accurate approximations (see [G, Bu1, Hen1, Hen2, JP1]), with particular emphasis on the set $E_{\{1,2\}}$. In this section we briefly outline the method first presented in [JP1, JP2].

The key idea is to exploit the fact that the transfer operator $\mathcal{L}_{A,s}$ preserves certain spaces of analytic functions such as the Hilbert Bergman space H defined in §2. As noted previously, the leading eigenvalue of $\mathcal{L}_{A,s} : H \to H$ is still $\lambda_A(s) = \exp(P_A(s))$. Crucially, however, this eigenvalue can be related to a zero of a certain analytic function; more precisely, it is the reciprocal of the zero z of minimum modulus of

$$\det(I - z\mathcal{L}_{A,s}) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\underline{i}\in A^n} \frac{w_{\underline{i}}^{2s}}{1-(-1)^n w_{\underline{i}}^2}\,, \tag{4.1}$$

where $w_{\underline{i}} := \prod_{r=0}^{n-1} T^r x_{\underline{i}}$ (see also Step 3 below).

In fact, as the notation suggests, $\det(I - z\mathcal{L}_{A,s})$ is the *determinant* for $\mathcal{L}_{A,s}$: its zeros are precisely the reciprocals of the non-zero eigenvalues of $\mathcal{L}_{A,s} : H \to H$ (the operator is in particular compact so these eigenvalues form a sequence converging to zero). This observation goes back to the fundamental paper [Ru1] of Ruelle, who first used these determinants to study certain dynamical zeta functions.

In fact for the special case at hand, where the underlying dynamical system is Gauss's continued fraction map, we have the following a priori information about the zeros z of any $\det(I - z\mathcal{L}_{A,s})$ in the case where s is real and positive:

Theorem 2. *If s is real and positive then the zeros z of $\det(I - z\mathcal{L}_{A,s})$ are all real.*

The proof of Theorem 2 is by [JGU, Thm. 2], where it is shown that $\mathcal{L}_{A,s}$ is conjugate to a self-adjoint operator (see also [Bab], [May] for analogous results when $A = \mathbb{N}$), and hence has real spectrum.

For the Hausdorff dimension $\dim(E_A)$ we are interested in the value of s for which the leading eigenvalue $\lambda_A(s)$ is equal to 1, so:

Theorem 3. *The Hausdorff dimension $\dim(E_A)$ is the smallest (in modulus) solution s of the equation*

$$\det(I - \mathcal{L}_{A,s}) = 0\,. \tag{4.2}$$

In fact analytic continuation of (4.1) defines an *entire* function of z. The consequent rapid decay of its Taylor coefficients ensures that it is well approximated by its sequence of Taylor polynomials. Each such polynomial is computable in terms of finitely many quadratic irrationals $x_{\underline{i}}$, thereby suggesting the following algorithmic scheme for approximating solutions to (4.2):

Step 0. Fix some $N \in \mathbb{N}$; choosing N larger results in a better approximation to $\dim(E_A)$, but prolongs the computational time needed[1].
Step 1. Compute all strings $\underline{i} \in A^n$, for $1 \leq n \leq N$.
Step 2. Compute the quadratic irrationals $x_{\underline{i}} = [0; i_1, \ldots, i_n, i_1, \ldots, i_n, \ldots]$ to some given numerical tolerance, for all $\underline{i} \in A^n$, $1 \leq n \leq N$. For example in Mathematica^{TM}, version 4 onwards, the $x_{\underline{i}}$ can be computed from the strings $\underline{i}$ using the FromContinuedFraction command.
Step 3. For $1 \leq n \leq N$ and $\underline{i} \in A^n$, define $w_{\underline{i}} = \prod_{r=0}^{n-1} T^r x_{\underline{i}}$, where $T(x) = 1/x \pmod 1$ is the Gauss map.
Step 4. For $1 \leq n \leq N$ and $\underline{i} \in A^n$, define

$$F_{\underline{i}}(s) = \frac{w_{\underline{i}}^{2s}}{1 - (-1)^n w_{\underline{i}}^2} .$$

Step 5. For each $1 \leq n \leq N$, set $c_n(s) = \frac{1}{n} \sum_{\underline{i} \in A^n} F_{\underline{i}}(s)$.
Step 6. Define coefficients $d_m(s)$ by the equation

$$1 + \sum_{m=1}^{\infty} d_m(s) z^m = \exp - \sum_{n=1}^{\infty} c_n(s) z^n ,$$

where we regard z as a formal variable (though in fact the expression defines an entire function of z). Each $d_m(s)$ is defined solely in terms of $c_1(s), \ldots, c_m(s)$:

$$d_m(s) = \sum_{\substack{(m_1, \ldots, m_j) \\ m_1 + \cdots + m_j = m}} \frac{(-1)^j}{j!} \prod_{l=1}^{j} c_{m_l}(s) .$$

Step 7. Define the function $\Delta_N(s) := 1 + \sum_{m=1}^{N} d_m(s)$.
Step 8. Locate the largest real zero s_N of Δ_N by some appropriate numerical method (e.g. the FindRoot command in Mathematica^{TM}); s_N is the required approximate value of $\dim(E_A)$.

The convergence $s_N \to s_\infty = \dim(E_A)$ is extremely fast: it is $O(\theta^{N^2})$ for some $0 < \theta < 1$ (see [JP1]). Computing s_{N+1} and comparing it to s_N gives a quick empirical means of checking how close the approximation is to the true Hausdorff dimension, though rigorous error bounds are also possible (see [JP2]).

[1]For $N = \infty$ we recover the *exact* value of $\dim(E_A)$, but the required computations cannot be performed in finite time.

5. Evidence for the Texan conjecture

In this final section we investigate whether there is any evidence for the Texan conjecture, by numerically computing the Hausdorff dimension of various bounded type sets E_A. Of course these computations are only approximate, but are reliable up to the accuracy given, which is typically four or five significant figures. The approximations were obtained by using the algorithm of §4, with $N = 4$ and 5.

Our main conclusion is that the conjecture is to be believed: extensive computations gave no hint of any "gaps" in the closure of $\mathcal{D}$. On the basis of Table 1, combined with Theorem 1, we have the more modest:

Theorem 4. *$\mathcal{D}$ is $\frac{1}{50}$-dense in $[0,1]$.*

The value $\frac{1}{50}$ is rather arbitrary, and was chosen so as to keep Table 1 to a reasonable size; our calculations indicate that it can be replaced by a much smaller (and conjecturally arbitrarily small) value.

A	Dimension
$\{1,4,7\}$	$0\cdot5178$
$\{1,2\}$	$0\cdot5312$
$\{1,3,8\}$	$0\cdot5438$
$\{1,3,7\}$	$0\cdot5532$
$\{1,3,6\}$	$0\cdot5652$
$\{1,3,5\}$	$0\cdot5813$
$\{1,2,10\}$	$0\cdot5951$
$\{1,3,4\}$	$0\cdot6042$
$\{1,2,7\}$	$0\cdot6179$
$\{1,2,7,40\}$	$0\cdot6265$
$\{1,2,5\}$	$0\cdot6460$
$\{1,2,5,40\}$	$0\cdot6532$
$\{1,2,4\}$	$0\cdot6692$
$\{1,2,4,40\}$	$0\cdot6754$
$\{1,2,4,15\}$	$0\cdot6899$
$\{1,2,3\}$	$0\cdot7056$
$\{1,2,4,7\}$	$0\cdot7185$
$\{1,2,4,6\}$	$0\cdot7275$
$\{1,2,4,5\}$	$0\cdot7400$
$\{1,2,3,6\}$	$0\cdot7588$
$\{1,2,3,5\}$	$0\cdot7709$
$\{1,2,3,4\}$	$0\cdot7889$
$\{1,2,3,4,10\}$	$0\cdot8081$
$\{1,2,3,4,6\}$	$0\cdot8269$
$\{1,2,3,4,5\}$	$0\cdot8368$
$\{1,2,3,4,5,9\}$	$0\cdot8541$
$\{1,2,3,4,5,7\}$	$0\cdot8616$
$\{1,2,3,4,5,6\}$	$0\cdot8676$
$\{1,2,3,4,5,6,8\}$	$0\cdot8851$
$\{1,2,3,4,5,6,7\}$	$0\cdot8889$
$\{1,\dots,8\}$	$0\cdot9045$
$\{1,\dots,9\}$	$0\cdot9164$
$\{1,\dots,10\}$	$0\cdot9257$
$\{1,\dots,13\}$	$0\cdot9445$
$\{1,\dots,18\}$	$0\cdot961$
$\{1,\dots,34\}$	$0\cdot980$

Table 1: Approximate[2] Hausdorff dimensions $\dim(E_A)$.

[2]With the exception of the final two entries, the approximate dimensions listed in Table 1 give the correct first 4 digits in the full decimal expansion of $\dim(E_A)$; in particular we do not round up the final digit. The final two entries are correct to 3 decimal places.

As a further test of the plausibility of the Texan conjecture, we now consider the more accurate approximation (to within 1/1000, say) of various values $d \in (1/2, 1)$ by dimensions $\dim(E_A)$. As in the proof of Theorem 1, the approximation of d will be from *below*. That is, we seek a finite sequence of increasing digit sets $A_1 \subset A_2 \subset \ldots \subset A_m$ such that $d - \dim(E_{A_m}) < 1/1000$. The A_i will be determined by a natural greedy algorithm: A_i has cardinality $i+1$, and its dimension equals $\sup\{\dim(E_A) : |A| = i+1\} \cap [0, d]$.

Somewhat arbitrarily, let us successively consider the four choices $d = 0 \cdot 6$, $d = 0 \cdot 7$, $d = 0 \cdot 8$, and $d = 0 \cdot 9$.

The case $d = 0 \cdot 6$: We start with the digit set $A_1 = \{1, 2\}$. The dimension $\dim(E_{A_1})$ is approximately $0 \cdot 53128$. In fact the dimension of $E_{\{1,2\}}$ is a favourite test case for the efficiency of numerical methods (see [G, Bu1, Hen1, Hen2, JP1]), and is already known with high precision. We take the opportunity here to improve on the approximation given in [JP1] by asserting:

Theorem 5. *The approximation*

$$dim(E_{\{1,2\}}) = 0 \cdot 531280506277205141624468647368471785493059109018398779\ldots$$

is accurate to 54 decimal places.

The above estimate is obtained by the algorithm of §4, using all quadratic irrationals $x_{\underline{i}}$ for which $\underline{i}$ is of length at most 16 to construct an approximation of $\det(I - \mathcal{L}_{\{1,2\},s})$ (see Figure 1). (In fact the estimate of [JP1], which was accurate to 25 decimal places, used the same set of $x_{\underline{i}}$, but for obscure reasons employed an algorithm involving $\det(I - \mathcal{L}^2_{\{1,2\},s})$ rather than $\det(I - \mathcal{L}_{\{1,2\},s})$).

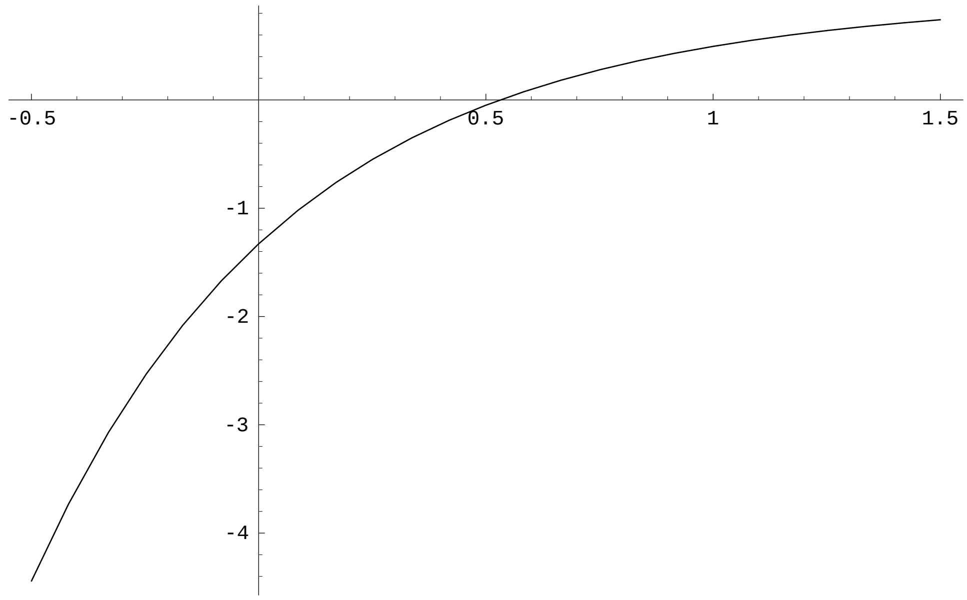

FIGURE 1. GRAPH OF $s \mapsto \det(I - \mathcal{L}_{\{1,2\},s})$

Fixing $s = \dim(E_{\{1,2\}})$, the other zeros of $\det(I - z\mathcal{L}_{\{1,2\},s})$ are of ergodic theoretic significance for the measurable dynamical system $(T, E_{\{1,2\}}, \mu)$, where μ denotes the unique T-invariant probability measure on $E_{\{1,2\}}$ which is equivalent to s-dimensional Hausdorff measure. The first few of these zeros are, in order of increasing modulus: 1, $-3 \cdot 20752$, $9 \cdot 12366$, $-24 \cdot 8796$, $66 \cdot 7626$, $-177 \cdot 753$, $471 \cdot 166 \ldots$. By Theorem 2 we know that *all* the zeros of $\det(I - z\mathcal{L}_{\{1,2\},s})$ are real; it seems likely that the zeros continue to alternate in sign, but we are unable to prove this. The zeros can be considered as *resonances* for the system $(T, E_{\{1,2\}}, \mu)$, and govern the exponential rates of mixing. In particular, if f, g are analytic in a sufficiently small complex neighbourhood of $[0,1]$ then $\int f\, g \cdot T^n \, d\mu \to \int f \, d\mu \int g \, d\mu$ at a rate which is $O\left(3.20752^{-n}\right)$ (the implicit constant in this asymptotic will depend on the norms of f and g).

Continuing with the approximation of $0 \cdot 6$ from below, we note that choosing $A_2 = \{1,2,3\}$ is no use, because $\dim(E_{\{1,2,3\}}) \approx 0 \cdot 70566$. In fact $\dim(E_{\{1,2,n\}}) > 0 \cdot 6$ for all $n \leq 9$, whereas $\dim(E_{\{1,2,10\}}) \approx 0 \cdot 59511$, so we take $A_2 = \{1,2,10\}$. Now A_3 will be a 4-digit set $\{1,2,10,n\}$, and the smallest n for which $\dim(E_{\{1,2,10,n\}}) < 0 \cdot 6$ is $n = 77$, therefore $A_3 = \{1,2,10,77\}$. In fact $\dim(E_{\{1,2,10,77\}}) \approx 0 \cdot 59997$ (whereas $\dim(E_{\{1,2,10,76\}}) \approx 0 \cdot 60005$), which is within $1/1000$ of $d = 0 \cdot 6$.

The case $d = 0 \cdot 7$*:* As before we start with the digit set $A_1 = \{1,2\}$, so that $\dim(E_{A_1}) \approx 0 \cdot 53128$. Next we choose $A_2 = \{1,2,4\}$, which gives $\dim(E_{A_2}) \approx 0 \cdot 66922$. Now $\dim(E_{\{1,2,4,10\}}) \approx 0{\cdot}7023$, whereas $\dim(E_{\{1,2,4,11\}}) \approx 0{\cdot}69892$, so we take $A_3 = \{1,2,4,11\}$. At the next step we are already within $1/1000$ of $0 \cdot 7$: taking $A_4 = \{1,2,4,11,119\}$ gives $\dim(E_{A_4}) \approx 0 \cdot 699993$, whereas $\dim(E_{\{1,2,4,11,118\}}) \approx 0 \cdot 700006$.

The case $d = 0 \cdot 8$*:* Let $A_1 = \{1,2\}$, $A_2 = \{1,2,3\}$ (with $\dim(E_{\{1,2,3\}}) \approx 0 \cdot 7056$), and $A_3 = \{1,2,3,4\}$ (with $\dim(E_{\{1,2,3,4\}}) \approx 0 \cdot 7889$). We next choose $A_4 = \{1,2,3,4,15\}$, with $\dim(E_{\{1,2,3,4,15\}}) \approx 0 \cdot 7998$, whereas $\dim(E_{\{1,2,3,4,14\}}) \approx 0 \cdot 8009$. Setting $A_5 = \{1,2,3,4,15,220\}$ then gives $\dim(E_{A_5}) \approx 0 \cdot 79998$, which is within $1/1000$ of $0 \cdot 8$, as required.

The case $d = 0 \cdot 9$*:* As before, the first few A_i are defined to equal $\{1,2,\ldots,i+1\}$, provided the corresponding dimension $\dim(E_{A_i})$ is less than $d = 0 \cdot 9$. In fact this is the case for $1 \leq i \leq 6$, since $\dim(E_{\{1,\ldots,7\}}) \approx 0 \cdot 88895$, but fails for $i = 7$ as $\dim(E_{\{1,\ldots,8\}}) \approx 0 \cdot 9045$. Defining $A_7 = \{1,2,3,4,5,6,7,10\}$ gives $\dim(E_{A_7}) \approx 0 \cdot 89996$, a sufficiently accurate approximation to $0 \cdot 9$.

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O. Jenkinson, School of Mathematical Sciences, Queen Mary, University of London, Mile End Rd., London E1 4NS, UK.

E-mail address: `omj@maths.qmul.ac.uk`