Synchronization: Exercises, problems, references

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LTCC

10-11 June 2010

Exercises

1. Introduction

1. Prove that the shortest reset word for the automaton A_n in the notes has length $(n-1)^2$.

2. Find the shortest reset word for the automaton in the "dungeon" example in the notes. Find all the elements in the monoid generated by this example.

2. Permutation groups

1. Find the orbits of the permutation group on $\{1, \ldots, 12\}$ generated by the two permutations $(1,2)(3,4)(5,7)(6,8)(9,11)(10,12)$ and $(1,2,3)(4,5,6)(8,9,10)$. (We use the convention that points omitted from the cycle notation are fixed, that is, in cycles of length 1.)

Where does this group fall in the hierarchy of properties defined in this section of the notes?

2. Let G be a transitive permutation group, and N a normal subgroup of G. Show that all the orbits of N have the same size.

3. Synchronizing and separating groups

1. Consider the group $G = A_5 \times A_5$ in its diagonal action as described in Notes 2, Example 6. Is it synchronizing?

2. Let us say that the permutation group G on Ω has *Property X* if there do *not* exist two non-trivial partitions π and ρ of Ω such that, if A is any part of π, and B any part of ρ, then $|Ag \cap B| = 1$. (This property is due to Peter M. Neumann.)

- Suppose that G is transitive, and π and ρ are witnesses that G does not have property X. Show that π and ρ are uniform.

- Show that a synchronizing group has property X, and that a group with property X is basic.
- Show that property X is closed upwards.

4. Graph homomorphisms

1. Let X be a vertex-transitive graph. Show that the number of vertices of $\mathrm{Core}(X)$ divides the number of vertices of X.

2. Give an example to show that the map from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(\mathrm{Core}(X))$ in the proof of Theorem 6 need not be a group homomorphism.

3. Prove that, if there is a homomorphism from X to Y, then $\omega(X) \leq \omega(Y)$ and $\chi(X) \leq \chi(Y)$.

5. Graphs and monoids

1. Let X be the path of length 3 (Example 7). Calculate the orders of $\mathrm{End}(X)$ and $\mathrm{End}(\mathrm{Hull}(X))$.

2. Prove directly that any endomorphism of the Petersen graph is an automorphism.

3. Recall the definition of Property X in the exercises to Section 3. Let G be a permutation group on Ω.

- Show that G does not have property X if and only if there exists a non-trivial graph X on Ω with $G \leq \mathrm{Aut}(X)$, which has the property that $\chi(X)\chi(\overline{X}) = |\Omega|$.
- Construct a group with property X which is not synchronizing, and a basic group which does not have property X.

4. Let $q(n)$ be the minimum, over all transitive but imprimitive groups G of degree n, of $|NS(G)|/n$. (By default we take $q(n) = 1$ if n is prime.) Prove that $\liminf_{n\to\infty} q(n) = 3/4$.

6. Examples

1. Let G be the group $S_2 \mathrm{\ Wr\ } S_5$, acting on the 16 diagonals of the 5-dimensional hypercube. (The vertices are the 32 sequences $(\epsilon_1, \ldots, \epsilon_5)$, where $\epsilon_i = \pm 1$; diagonals join opposite vertices $(\epsilon_1, \ldots, \epsilon_5)$ and $(-\epsilon_1, \ldots, -\epsilon_5)$.) Is G synchronizing?

2. Suppose that k divides n, and that $n > 2k$. Let G be the permutation group induced by the symmetric group S_n on the set of all partitions of $\{1, \ldots, n\}$ into n/k parts each of size k. Prove that G is non-synchronizing.[Hint: Baranyai's Theorem.]

7. Representation theory

1. Let $\mathbb{F}$ be a field. Let U be the submodule $\underline{\Omega}$ of the permutation module $\mathbb{F}\Omega$ for the symmetric group $G = \mathrm{Sym}(\Omega)$, and W the augmentation module. Show that

- U and W are the only non-trivial submodules of $\mathbb{F}\Omega$;
- if the characteristic of $\mathbb{F}$ divides $|\Omega|$, then $U \subseteq W$; otherwise $U \oplus W = \mathbb{F}\Omega$.

8. The infinite

1. Find the hull of a two-way infinite path.

2. Let X be an infinite graph. Suppose that $\chi(Y) \leq m$ for any finite subgraph Y of the infinite graph X, where m is a positive integer. Show that $\chi(X) \leq m$. [If you know the Compactness Theorem of propositional or first-order logic, this is the best way to do it.]

Open problems

1. Spreading and QI

The big problem to which we don't know the answer is:

> Is there a permutation group which is spreading but not QI?

Opinion is divided: I think there should be, others think not. Extensive computation has failed to find one.

2. Is spreading necessary?

We showed in the lectures that, if G is spreading, then for any map f which is not a permutation, $\langle G, f \rangle$ contains a reset word with at most $n-1$ occurrences of f. Is the "spreading" condition necessary for this conclusion? (Clearly "synchronizing" is necessary: without this, there may not be a reset word at all!)

3. More examples

For further families of basic primitive permutation groups, decide whether the groups are synchronizing, separating, or spreading.

4. Affine groups

Are "synchronizing" and "separating" equivalent for affine groups? Note that, if A and B witness that an affine group is non-separating, and B is an affine subspace, then the translates of B form a section-regular partition with section A, demonstrating that the group is not synchronizing either. There do exist cases where one of A and B is a subspace but the other is not; is there an example where neither is a subspace?

5. Non-synchronizing ranks

Is there a constant c such that a transitive permutation group G of degree n is primitive if and only if it has fewer than cn non-synchronizing ranks? Is it true that a primitive group of degree n has $o(n)$ (or maybe even $O(\log n)$) non-synchronizing ranks?

What can be said about a transitive group G with either $n-2 \in NS(G)$ or $3 \in NS(G)$? Is such a group imprimitive with finitely many exceptions? If so, is there a generalisation?

6. Infinite examples

Does there exist a permutation group (of uncountably infinite degree) which is synchronizing but not 2-set transitive?

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