Synchronization 4: Graph homomorphisms

Peter J. Cameron

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Homomorphisms of relational structures

A homomorphism of an algebraic structure is a map which preserves the operations of the algebra: for example, a group homomorphism satisfies $f(xy) = f(x)f(y)$.

Similarly, a homomorphism of a relational structure preserves the relations of the structure: if R is an n-ary relation, then $R(x_1, \ldots, x_n)$ implies $R(f(x_1), \ldots, f(x_n))$.

For example, a homomorphism of an ordered set $(X, \leq)$ is an order-preserving map, while a homomorphism of a strict order $(X, <)$ is strictly order-preserving.

Endomorphisms and automorphisms

A homomorphism from a structure X to itself is an *endomorphism* of X. If it is a bijection, and its inverse is also an endomorphism, then it is an *automorphism* of X.

The set $\mathrm{End}(X)$ of endomorphisms of X is closed under composition, and contains the identity map; in other words it is a *transformation monoid* on X (a submonoid of the full transformation monoid).

Similarly, the set $\mathrm{Aut}(X)$ of automorphisms of X is a *permutation group* on X, a subgroup of the symmetric group.

Graphs

We are mostly concerned with graphs.

A *graph* consists of a set of vertices, with a collection of edges, each edge being a 2-element set of vertices. In other words, unlike the digraphs which represent automata, our graphs are undirected and have no loops and no multiple edges. Until the last lecture, all graphs are finite.

We write $v \sim w$ to denote that $\{v, w\}$ is an edge.

Graph homomorphisms

A *homomorphism* from a graph X to a graph Y is thus a map f from the vertex set of X to the vertex set of Y such that, if $\{v, w\}$ is an edge of X, then $\{f(v), f(w)\}$ is an edge of Y. Note that, if $\{v, w\}$ is not an edge, then we do not specify what its image is; it may be a non-edge, or an edge, or even a single vertex.

An *endomorphism* of a graph X is a homomorphism from X to X.

An *automorphism* is a bijective endomorphism which also maps non-edges to non-edges; that is, whose inverse is also an endomorphism.

Theorem 1. *An endomophism of a (finite) graph is an automorphism if and only if it is a bijection.*

Proof. The forward implication is clear. Conversely, a bijective homomorphism cannot decrease the number of edges; so, if it maps a graph to itself, then non-edges must map to non-edges, else the number of edges would strictly increase. $\square$

Hom-equivalence and hom-order

We write $X \to Y$ to denote that there exists a homomorphism from X to Y.

Let X and Y be graphs. We say that X and Y are *hom-equivalent* if $X \to Y$ and $Y \to X$ hold; we write this as $X \equiv Y$. We denote the hom-equivalence class of X by $[X]$.

Now we set $[X] \leq [Y]$ if there is a homomorphism from X to Y. (This is independent of the choice of representatives of the equivalence classes.) This relation is a partial order on the set of hom-equivalence classes, called the *hom-order*.

Homomorphisms, cliques and colourings

Let K_m denote the *complete graph* on m vertices, the graph in which every 2-element subset of the vertex set is an edge.

If f is a homomorphism from K_m to X, then the image of f is a set of m vertices of X, any two of which are joined by an edge (that is, the induced subgraph is a complete graph). Such a set of vertices is called a *clique* of size m in X. The *clique number* $\omega(X)$ of a graph X is the size of the largest clique in X.

Thus,

$$K_m \to X \text{ if and only if } \omega(X) \geq m.$$

A (proper) *colouring* of a graph is an assignment of colours to the vertices so that adjacent vertices get different colours. The *chromatic number* $\chi(X)$ of X is the smallest number of colours required for a proper colouring.

A homomorphism f from X to K_m satisfies $(v \sim w) \Rightarrow (f(v) \neq f(w))$, so is a proper colouring with m colours, and conversely.

Thus,

$$X \to K_m \text{ if and only if } \chi(X) \leq m.$$

Now any graph X satisfies $\omega(X) \leq \chi(X)$, since the vertices of a clique must all get different colours in a proper colouring. Hence:

Theorem 2. *A graph X satisfies $X \equiv K_m$ if and only if $\omega(X) = \chi(X) = m$.*

This class of graphs will be very important in what follows.

Timetabling

Suppose that we have a set C of classes to timetable. Let R be the set of classrooms, and S the set of timetable slots. A timetable consists of a map f from C to $R \times S$ (assigning a room and time to each class). If some rooms are pre-booked at certain times, we use a subset of $R \times S$ instead.

Give both C and $R \times S$ the structure of complete graphs. Now requiring that f is a homomorphism ensures that we do not put two classes in the same room at the same time.

Now form a graph (in a different colour) on the vertex set C by joining two classes c_1 and c_2 by an edge if either some student is in both classes, or both classes are to be taught by the same teacher. We form the graph on $R \times S$, in which (r_1, s_1) and (r_2, s_2) are joined if and only if $s_1 \neq s_2$. The homomorphism requirement ensures that no student or teacher is expected to be in different classes at the same time.

Other requirements can be included. For example, if some rooms are too small to accommodate some classes, we take a unary relation which holds on all the large classes and on all the pairs (r, s) where r is a large room.

Constraint satisfaction

Let R be a relational structure with a fixed signature (that is, with named relations of prescribed arities). The *constraint satisfaction problem* or *CSP* based on R is the following decision problem:

Instance: A relational structure X of the same signature as R.

Problem: Is there a homomorphism from X to R?

We see that CSP includes graph colouring and timetabling problems among many other things.

The *Feder–Vardy conjecture* asserts that, for any R, either the CSP based on R is in P, or it is NP-complete. Several instances of this are known to be true.

This contrasts with the similar-looking problem where we require an isomorphism. The *graph isomorphism problem* is thought by some to be of complexity intermediate between P and NP-complete.

Of course, if P = NP, then all these conjectures collapse!

The core of a graph

Let X be a graph. A *core* of X, written $\mathrm{Core}(X)$, is defined to be a graph Y with the smallest number of vertices subject to $X \equiv Y$.

Note that, if Y is a core of X, then it is a core of any graph in $[X]$. In particular, $Y = \mathrm{Core}(Y)$ is its own core. (We will say that a graph Y is a *core* if $\mathrm{Core}(Y) = Y$.)

We now show that the core of a graph is unique up to isomorphism, and give a test to recognise a core.

Theorem 3.

- *Any endomorphism of a core is an automorphism.*
- *Two cores which are hom-equivalent are isomorphic.*

Proof. (a) Let Y be a core and f an endomorphism of Y. Then there is a homomorphism from $f(Y)$ to Y, namely, the identity map on the vertices of $f(Y)$; so $Y \equiv f(Y)$. Since Y is a core, f must be onto, hence an automorphism.

(b) Similar. □

So now we can talk about *the* core of a graph.

Theorem 4. *The core of a graph X is K_m if and only if $\omega(X) = \chi(X) = m$.*

This is immediate from the theory developed above.

The class of graphs whose core is complete (those whose clique number and chromatic number are equal) is important in the theory of synchronization, as we will see.

In particular, for $m = 2$, we have the class of bipartite graphs (containing at least one edge).

Induced subgraphs and retractions

Y is an *induced subgraph* of X if its vertices are some of the vertices of X, and its edges are all the edges of X contained within the vertex set of Y.

A *retraction* of X is a homomorphism from X to a induced subgraph Y which acts as the identity on Y.

Theorem 5. *Let $Y = \mathrm{Core}(X)$. Then, up to isomorphism, Y is an induced subgraph of X, and there is a retraction of X onto Y.*

Proof. Let $Y = \mathrm{Core}(X)$. There is a homomorphism from Y to X, which is an isomorphism to its image; so we can identify Y with this image.

Now let f be a homomorphism from X to Y. The restriction of f to Y is an automorphism of Y; following f by the inverse of this automorphism, we get a retraction of X onto Y. □

Recognising cores

A graph X is a core if and only if all its endomorphisms are automorphisms. (We saw the forward implication. In the reverse direction, if X is not a core, then its core is a proper induced subgraph, and there is an endomorphism onto this subgraph.)

But this is hard to test. The decision problem for cores is:

Instance: A graph X.

Problem: Is X a core?

This is known to be NP-complete.

Cores of vertex-transitive graphs

A graph is *vertex-transitive* if its automorphism group acts transitively on its vertex set.

Theorem 6. *The core of a vertex-transitive graph is vertex-transitive.*

Proof. Let X be vertex-transitive. Let ι be the embedding of $Y = \mathrm{Core}(X)$ into X, and ρ the retraction of X onto Y. Let v and w be vertices of Y, and choose an automorphism g of X mapping v to w. Then $\iota g \rho$ is a homomorphism from Y to Y (necessarily an automorphism of Y, since Y is a core) which maps v to w. □

Other kinds of transitivity

Analogously, we can say that a graph is *edge-transitive*, or *non-edge transitive*, or indeed transitive on any kind of configuration we choose.

The same proof as above shows that if X possesses any kind of transitivity, then so does $\mathrm{Core}(X)$ (provided only that $\mathrm{Core}(X)$ possesses configurations of the appropriate type – for example, X might be a non-edge transitive graph whose core is complete).

In the next lecture, we will see that a much stronger result holds for non-edge transitive graphs: either such a graph is itself a core, or its core is complete.