Chapter 1 solutions

1.1. $A \subseteq B$ means that every element of A is in B, and $B \subseteq A$ that every element of B is in A. The combination of these two statements is just what is meant by $A = B$.

1.2. (a) In the expression $|A| + |B|$, we have counted every element of $A \cup B$, but elements of $A \cap B$ have been counted twoce; so we must subtract $|A \cap B|$ to get the right answer for $|A \cup B|$.

(b) There are $|A|$ choices for the first element of an ordered pair and (independently) B choices for the second, giving $|A| \cdot |B|$ pairs altogether.

1.3.

- None of the axioms: $\{(1,2)\}$.
- (E1) only: $\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$.
- (E2) only: $\{(1,2),(2,1)\}$.
- (E3) only: $\{(1,2),(2,3),(1,3)\}$.
- (E1) and (E2): $\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$.
- (E1) and (E3): $\{(1,1),(2,2),(3,3),(1,2)\}$.
- (E2) and (E3): The empty relation.
- All three: $\{(1,1),(2,2),(3,3)\}$.

Many other answers are possible. There are $2^9 = 512$ sets of ordered pairs, each of which must fit into one of these eight categories!

No relation on $\{1,2\}$ can be reflexive and symmetric but not transitive, for example.

1.4. (a) It is reflexive, symmetric and transitive, since it contains all possible ordered pairs. (The conclusion of each of the three laws is that a certain ordered pair is in R.) The corresponding partition has just a single part, namely the whole set A.

(b) This relation R is trivially reflexive and symmetric. If $(a,b) \in R$ and $(b,c) \in R$, then $a = b$ and $b = c$, so $a = c$ and hence $(a,c) \in R$. Thus it is also transitive. The corresponding partition has the property that each of its parts consists of a single element.

1.5. Yes. There is one equivalence relation on the empty set, namely the empty set of ordered pairs; and one partition, the partition with the empty set of parts.

1.6. From $\{1,2,3\}$ to $\{1,2,3,4\}$: $4^3 = 64$ functions of which $4 \cdot 3 \cdot 2 = 24$ are one-to-one and none are onto.

From $\{1,2,3,4\}$ to $\{1,2,3\}$: $3^4 = 81$ functions, of which none are one-to-one and 36 are onto.

1.7. (a) Not a function because $F(0)$ is not defined. We can fix it in either of two ways: either say that F is a function from $\mathbf{R} \setminus \{0\}$ to $\mathbf{R}$, or else *define* the value of $F(0)$ to be anything at all, say 0.

(b) Not a function because the two roots c and d of the quadratic can be written in either order, and we have not given a rule to specify which order we are using. (For example, the roots of the quadratic $x^2 - 3x + 2 = 0$ are 1 and 2, so $F(-3,2)$ is either $(1,2)$ or $(2,1)$, but we haven't said which.) We could fix this by specifying an order. For example, we could take c to be the root with smaller real part; if they have the same real parts, take c to be the one with smaller imaginary part. (If both the real and the imaginary parts are equal, then $c = d$, and the problem doesn't arise.) Alternatively, we could define $F(a,b)$ to be the *set* $\{c,d\}$, in which case F maps $\mathbf{C}^2$ to the set of subsets of $\mathbf{C}$ containing at most two elements.

1.8 It is easier (and equivalent) to show that there are exactly five partitions of $\{1,2,3\}$. They are the partition with a single part, the partition with all parts containing a single element, and three others having a part of size 1 and one of size 2, namely $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{3\},\{1,2\}\}$.

On a set of four points there are 15 partitions: one with a single part, four with parts of sizes 1 and 3, three with parts of sizes 2 and 2, six with parts of sizes 1, 1 and 2, and one with all its parts of size 1.

1.9. (a) Reflexive and transitive, not symmetric.

(b) Assuming that every capital city in Europe has a railway station, this is an equivalence relation. The equivalence classes change over time, since both the set of capital cities and the rail connections vary.

(c) Reflexive and transitive, not symmetric.

(d) An equivalence relation. There are four equivalence classes, the congruence classes modulo 4.

1.10. Recall that x and y are in the same equivalence class of $\mathrm{KER}(f)$ if and only if $f(x) = f(y)$. So, defining a function F from the set of equivalence classes to $\mathrm{Im}(f)$ by the rule that $F(C) = f(x)$ for some $x \in C$, we see that F is well-defined. It is clearly onto $\mathrm{Im}(f)$. If $F(C_1) = F(C_2)$, then elements of C_1 and C_2 are equivalent, and so $C_1 = C_2$; so F is one-to-one.

1.11. (a) We check that the appropriate laws hold.

- For any x, we have $x \sim x$ and $x \sim x$ (as we're given that $\sim$ is reflexive), so $x \equiv x$.
- Suppose that $x \equiv y$. Then $x \sim y$ and $y \sim x$; so $y \equiv x$.
- Suppose that $x \equiv y$ and $y \equiv z$. Thus $x \sim y$, $y \sim x$, $y \sim z$ and $z \sim y$. From $x \sim y$ and $y \sim z$ and the transitivity of $\sim$ we infer that $x \sim z$. Similarly $z \sim x$. So $x \equiv z$.

(b) We have $x \sim y$, $x \sim x_1$, $x_1 \sim x$, $y \sim y_1$ and $y_1 \sim y$. Applying the transitive law twice to the third, first and fourth of these relations shows that $x_1 \sim y_1$.

1.12. The truth table is as follows:

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \vee (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The fact that every entry in the last column is T shows that the formula is logically valid.

1.13. $x \in A \triangle B$ if and only if either p is true and q is false, or p is false and q is true. These are just the cases where the truth table for $\neg(p \Leftrightarrow q)$ has entries T.

The entries T in the truth table for $p \Rightarrow q$ occur in all rows except that where p is true and q false. So the set represented is $(A \setminus B)'$, the complement of $A \setminus B$.

1.14. The corresponding expressions are $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$. Construct a truth table to show that these expressions are logically equivalent.

1.15. (a) Follow the argument in the text for $\sqrt{2}$, replacing 2 by p. Wee need to know that, if p divides x^2, then p divides x. This follows from the Fundamental Theorem of Arithmetic. For, if p doesn't divide x, then it doesn't occur in the prime factorisation of x, and hence not in the prime factorisation of x^2 either.

(b) Suppose that $\sqrt[3]{2} = x/y$, where x and y are positive integers, and the fraction is in its lowest terms. Then we have $x^3 = 2y^3$. Thus, x^3 is even, and so also x is even. [Why?] Say $x = 2u$. Then $8u^3 = 2y^3$, and so $4u^3 = y^3$. So y^3, and hence also y, is even, contrary to the assumption that the fraction is in its lowest terms.

1.16. All that requires clarification is the notion that a rational number x has a unique fractional part. Suppose that $x = a/b$, with $b > 0$. The Division Algorithm gives $a = bq + r$, with $0 \leq r < b$. Thus $x = q + (r/b)$, with $0 \leq (r/b) < 1$. Hence q is the integer part, and (r/b) the fractional part. If there were two different expressions for x, then the integer parts would differ by an integer, while the fractional parts would differ by less than one; this is impossible.

1.17. (a) This is true for $n = 1$. Assume that it holds for a value n. Adding on $n+1$, we find that the sum of the first $n+1$ integers is $n(n+1)/2 + (n+1) = (n+1)(n+2)/2$; so the result holds also for $n+1$.

(b) Use the result of (a): what we are asked to show is that the sum of the cubes of the first n positive integers is $n^2(n+1)^2/4$. Now prove this by induction as in the earlier examples.

1.18. For $n = 1$, we interpret A^1 as being just A, and so the induction starts correctly.

Suppose that $|A^n| = |A|^n$. Then $A^{n+1} = A^n \times A$, and so

$$|A^{n+1}| = |A^n| \cdot |A| = |A|^n \cdot |A| = |A|^{n+1}.$$

This verifies the result for $n+1$, the inductive step. So it is true for all n by induction.

1.19. Suppose that such a sequence exists. By the Well-ordering Principle, it has a least element, say a_k. But then the facts that $a_k \leq a_{k+1}$ (since a_k is the least element) and $a_k > a_{k+1}$ (given) conflict.

1.20. $P(1)$ is indeed true, but $P(2)$ is clearly false (two horses don't necessarily have the same ccolour). So the inductive step must fail for $n = 1$. Indeed, given a set $\{H_1, H_2\}$ of two horses, it is indeed true that all the horses in $\{H_1\}$ ave the same colour, and all the horses in $\{H_2\}$ have the same colour; but these sets don't overlap, so no conclusion can be drawn.

1.21. Let us just consider the term of degree 2 in the product of the polynomials $f(x) = a_0 + a_1x + a_2x^2 + \dots$ and $g(x) = b_0 + b_1x + b_2x^2 + \dots$. In the product fg, this coefficient is $(a_0b_2 + a_1b_1) + a_2b_0$, while in gf it is $(b_0a_2 + b_1a_1) + b_2a_0$. To show that these two expressions are equal, we need that $a_0b_2 = b_2a_0$, $a_1b_1 = b_1a_1$, and $a_2b_0 = b_0a_2$ (commutativity of multiplication), and that $(u+v)+w = (w+v)+u$ (which requires associativity and commutativity of addition to do in three steps):

$$(u+v)+w = u+(v+w) = (v+w)+u = (w+v)+u.$$

1.22 (a) True; (b) False (should be $(fg)(t) = f(t)g(t)$).

Proof of (a): if $f(x) = \sum a_nx^n$ and $g(x) = \sum b_nx^n$, then

$$(f+g)(t) = \sum(a_n+b_n)x^n = \sum a_nx^n + \sum b_nx^n,$$

using the commutative and associative properties of the coefficients to rearrange the equationds. The proof of (b) is similar.

1.23 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, and $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$, then calculation shows that

$$(A+B)C = \begin{pmatrix} (a+e)i+(b+f)k & (a+e)j+(b+f)l \\ (c+g)i+(d+h)k & (c+g)j+(d+h)l \end{pmatrix},$$

$$AC+BC = \begin{pmatrix} (ai+bk)+(ei+fk) & (aj+bl)+(ej+fl) \\ (ci+dk)+(gi+hk) & (cj+dl)+(gj+hl) \end{pmatrix}$$

So the equality requires four calculations of which the first is typical:

$$(a+e)i+(b+f)k = (ai+ei)+(bk+fk) = (ai+bk)+(ei+fk),$$

where the first equality uses the distriutive law, and the second the commutative and associative laws for addition.

1.24. No coincidence. The sum of the diagonal elements of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is $ae+bg+cf+dh$; for $\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the sum is $ea+fc+gb+hd$, which is the same.

1.25 Let $A = (a_{ij})$ and $B = (b_{ij})$, where $a_{ij} = b_{ij} = 0$ for $i > j$. Then $A+B = C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$; clearly $c_{ij} = 0$ for $i > j$. Also $AB = D = (d_{ij})$, where

$$d_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

If $i > j$ then, for any value of k, necessarily either $i > k$ or $k > j$. (Otherwise $i \leq k$ and $k \leq j$, whence $i \leq j$, contrary to assumption.) So each term in the sum has either $a_{ik} = 0$ or $b_{kj} = 0$, and hence $d_{ij} = 0$. Thus both $A+B$ and AB are upper triangular.

1.26. If $a = c$ and $b = d$, then $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$. The converse is more difficult. So suppose that $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$. Since two sets are equal if and only if they have the same members, this implies that *either*

- $\{a\} = \{c\}$, $\{a,b\} = \{c,d\}$, *or*
- $\{a\} = \{c,d\}$, $\{a,b\} = \{c\}$.

In the first case, we have $a = c$, and then *either* $a = c$ and $b = d$, *or* $a = d$ and $b = c$. The first subcase is exactly what we want. In the second subcase, we have $c = a = d = b$, so all four elements are equal. In the second itemized case, we have $c = a = d$ amd $a = c = b$, so again all four are equal.