Solutions to Exercises
Chapter 10: Ramsey's Theorem

1 A platoon of soldiers (all of different heights) is in rectangular formation on a parade ground. The sergeant rearranges the soldiers in each row of the rectangle in decreasing order of height. He then rearranges the soldiers in each column in decreasing order of height. Using the Pigeonhole Principle, or otherwise, prove that it is not necessary to rearrange the rows again; that is, the rows are still in decreasing order of height.

Let a_{ij} denote the height of the soldier in row i and column j after the first rearrangement (within the rows). Thus, the numbers $a_{i1}, a_{i2}, \ldots, a_{in}$ form a decreasing sequence.

Let b_{ij} denote the height of the soldier in row i and column j after the second rearrangement. Then the numbers $b_{1j}, b_{2j}, \ldots, b_{nj}$ form a decreasing sequence, and in fact are the numbers $a_{1j}, a_{2j}, \ldots, a_{mj}$ arranged in decreasing order.

We have to prove that the rows are also decreasing; that is, that $b_{ij} > b_{i\,j+1}$ for all i and j. Suppose not, and suppose that $b_{ij} < b_{i\,j+1}$ for some i and j. Choose a number x such that $b_{ij} < x < b_{i\,j+1}$. Now b_{ij} is the ith largest of the numbers in the set $A_j = \{a_{1j}, \ldots, a_{mj}\}$, while $b_{i\,j+1}$ is the ith largest number in the set $A_{j+1} = \{a_{1\,j+1}, \ldots, a_{m\,j+1}\}$. So at least i members of A_{j+1} are greater than x, whereas fewer than i members of A_j are greater than x.

But we can pair off the elements of A_j and A_{j+1} so that the member of A_j is always the larger of the pair: for we have $a_{kj} > a_{k\,j+1}$ for all k. So there must be at least as many members of A_j greater than x than there are in A_{j+1}, a contradiction.

2 Show that any finite graph contains two vertices lying on the same number of edges.

Let G be a graph with n vertices $v_1, \ldots, v_n$. Place v_i in a pigeonhole labelled m if it has exactly m neighbours. The possible labels on the pigeonholes are $0, 1, 2, \ldots, n-1$. So n vertices are placed in n pigeonholes. However, the pigeonholes labelled 0 and $n-1$ cannot both be occupied. For, if v_i were in pigeonhole zero and v_j in pigeonhole $n-1$, then v_i is joined to no other vertices (in particular, not to v_j), but v_j is joined to all other vertices (in particular, to v_i), a contradiction. Now the n vertices lie in at most $n-1$ pigeonholes, and so some pigeonhole contains at least two vertices.

3 (a) Show that, given five points in the plane with no three collinear, the number of convex quadrilaterals formed by these points is odd.
(b) Prove Fact 2 in the proof of (10.5.2).

(a) There are several cases.

First, suppose that the five vertices form a convex pentagon. Then any four of them form a convex quadrilateral; so there are five convex quadrilaterals.

Next, suppose that four points form a convex quadrilateral (say $p_1, p_2, p_3, , p_4$ in anti-clockwise order) which contains the fifth in its interior. Let q be the intersection of the diagonals p_1p_3 and p_2p_4. Then p_5 lies on neither diagonal, so is in one of the four triangles into which they dissect the quadrilateral, say qp_1p_2. Then the quadrilaterals $p_1p_3p_4p_5$ and $p_2p_3p_4p_5$ are convex but the other two are not; so there are three convex quadrilaterals in all.

Finally, suppose that three points (say p_1, p_2, p_3) form a triangle with p_4 and p_5 in its interior. The line p_4p_5 meets two of the three sides, say p_1p_2 and p_1p_3. Then $p_2p_3p_4p_5$ is the only convex quadrilateral.

(b) Suppose that any four points form a convex quadrilateral, but the entire set is not a convex n-gon.

Consider a line passing through p_n and rotating around this point. Suppose its starting position is the line p_np_1. As it rotates, it will successively pass through all the other points, and after rotating through π, it passes again through p_1. We record whether the points lie on the same end of the line as p_1 originally did. After rotating through π, the point p_1 is on the other end of the line. So it must occur that two successive points, say p_i and p_j, lie on opposite ends.

It cannot occur that all the other points lie in the sector bounded by the lines p_np_i and p_np_j. For this sector is convex, so the convex hull of the points would be contained in the sector, and would not contain p_n. By construction, no point lies in the sector between p_ip_n produced beyond p_n and the line p_np_j; or in the sector between p_jp_n produced and p_np_i. So there is a point p_k in the sector between p_ip_n produced and p_jp_n produced. Then p_n lies in the convex hull of $p_ip_jp_k$, and the quadrilateral $p_ip_jp_kp_n$ is not convex.

4 Show that, if $N > mnp$, then any sequence of N real numbers must contain either a strictly increasing subsequence with length greater than m, a strictly decreasing subsequence with length greater than n, or a constant subsequence of length greater than p. Show also that this result is best possible.

Let N real numbers be given, with $N > mnp$. By the Pigeonhole Principle, if only mn or fewer distinct values occur, then some value must be taken by more than p numbers, so there is a constant subsequence of length greater than p. Otherwise, more than mn values occur, and the second proof of (10.5.1) shows that there is either a strictly increasing sequence of length greater than m, or a strictly decreasing sequence of length greater than n.

To show that the result is best possible, take the sequence on page 155, and repeat each term p times.

5 (a) Show that any infinite sequence of real numbers contains an infinite subsequence which is either constant or strictly monotonic.

(b) Using the Principle of the Supremum, prove that every increasing sequence of real numbers which is bounded above is convergent.

(c) Hence prove the *Bolzano–Weierstrass Theorem*: Every bounded sequence of real numbers has a convergent subsequence.

(a) Given an infinite sequence $x_1, x_2, \ldots$, colour the pair $\{i, j\}$ of positive integers (where $i < j$) red if $x_i < x_j$, green if $x_i > x_j$, and blue if $x_i = x_j$. By Ramsey's Theorem, there is an infinite monochromatic subset, which indexes an infinite monotone or constant subsequence of the original sequence.

(b) Suppose that $y_1, y_2, \ldots$ is an increasing sequence of real numbers which is bounded above. Let l be its supremum, or least upper bound. We claim that l is the limit of the sequence. Let $\varepsilon > 0$ be given. Then $l - \varepsilon$ is not an upper bound, so some term y_n of the series exceeds it. Since the series is increasing, we have $y_m > l - \varepsilon$ for all $m \geq n$. But $l + \varepsilon$ is a strict upper bound, so $y_m < l + \varepsilon$ for all m. Combining these two statements, we see that $|y_m - l| < \varepsilon$ for all $m \geq n$, where n was calculated from ε. Thus, by definition, l is the limit of the sequence, which is therefore convergent.

(c) Let $x_1, x_2, \ldots$ be a bounded sequence of real numbers By (a), there is a constant or monotone subsequence. A constant subsequence is trivially convergent; an increasing subsequence was shown to be convergent in part (b); and dually a decreasing subsequence is convergent. So the Bolzano–Weierstrass Theorem is proved.

6 (a) Let X be the set of residues modulo 17. Colour the 2-element subsets of X by assigning to $\{x,y\}$ the colour red if

$$x-y \equiv \pm 1, \pm 2, \pm 4 \text{ or } \pm 8 \pmod{17},$$

blue otherwise. Show that there is no monochromatic 4-set. [HINT: By symmetry, we may assume that the 4-set contains 0 and 1; this greatly reduces the number of cases to be considered!]

(b) Find a colouring of the 2-subsets of an 8-set red and blue so that there is no red 3-set and no blue 4-set.

(c) Let X consist of all subsets of $\{0,1,2,3,4\}$ of even cardinality. Colour the 2-subsets $\{x,y\}$ of X as follows:

- red if $x \triangle y = \{i,j\}$ and $i-j \equiv \pm 1 \pmod 5$;
- blue if $x \triangle y = \{i,j\}$ and $i-j \equiv \pm 2 \pmod 5$;
- green if $|x \triangle y| = 4$.

Show that there is no monochromatic 3-set.

(a) Note, as in the solution on page 342, that the operation of adding a fixed element of $\mathrm{GF}(17)$ doesn't change the colour of an edge (since $(y+a)-(x+a) = y-x$). Also, the values $\pm 1, \pm 2, \pm 4, \pm 8$ are precisely the quadratic residues in $\mathrm{GF}(17)$. Hence multiplication by a fixed residue doesn't change the colour of an edge. (We have $ry - rx = r(y-x)$; and, by Exercise 13 of Chapter 9, if r is a residue, then $r(y-x)$ is a residue if and only if $y-x$ is a residue.) Similarly, if n is a non-residue, then $n(y-x)$ is a residue if and only if $y-x$ is a non-residue; so multiplication by n changes the colour of every edge.

Suppose that there is a red 4-clique. If a is one of its points, then adding $-a$ gives a red 4-clique containing 0. Then, if b is a non-zero element, then b is a residue, and multiplication by the residue b^{-1} gives a 4-clique containing 0 and 1. The other two points are joined to both 0 and 1 by red edges, and hence are two of the three such points $2, 9, 16$. But any two of these are joined by blue edges. So no red 4-clique can exist.

If a blue 4-clique exists, then multiplication by a non-residue transforms it into a red 4-clique, an impossibility. So no blue 4-clique can exist either.

(b) Take the eight points joined to 0 by red edges in the example in part (a). This set can contain no blue 4-set, and no red 3-set (since such a set, together with 0, would form a red 4-set in the configuration of (a)).

(c) First we check that the definition is exhaustive. If $|A|$ and $|B|$ are even, then $|A \triangle B| = |A| + |B| - 2|A \cap B|$ is also even. Also, $(A \triangle X) \triangle (B \triangle X) = A \triangle B$, so the operation of taking the symmetric difference with a fixed set preserves the colours of edges. Thus, if there is a monochromatic 3-set, then there is one containing $\emptyset$. Thus, it suffices to prove that, *among the points joined to $\emptyset$ by red edges, all the edges are green and blue*, and similarly for the other colours. Furthermore, the permutation induced by shifting $0,1,2,3,4$ cyclically also preserves the colours.

The red neighbours of $\emptyset$ are $01, 12, 23, 34, 40$ (where 01 is short for $\{0,1\}$, and so on). Now the edges joining 01 to 12 and 04 are both blue (since, for example, $01 \triangle 12 = 02$), while those joining it to 23 and 34 are both green. Similarly for the other colours.

7 (a) Prove the following theorem of Schur:

> **Schur's Theorem** *There is a function f on the natural numbers with the property that, if the numbers $\{1,2,\ldots,f(n)\}$ are partitioned into n classes, then there are two numbers x and y such that x, y and $x+y$ all belong to the same class.*

(In other words, the numbers $\{1,2,\ldots,f(n)\}$ cannot be partitioned into n 'sum-free sets'.)

(b) State and prove an infinite version of Schur's Theorem.

(a) Take N to be one less than the Ramsey number $R(n,2,3)$. Now suppose that the set $\{1,2,\ldots,N\}$ is partitioned into n classes $C_1,\ldots,C_n$. Colour the 2-subsets of $\{1,2,\ldots,N+1\}$ with n colours $c_1,\ldots,c_n$ by the following rule: the 2-set $\{x,y\}$ (with $x<y$) has colour c_i if (and only if) $y-x \in C_i$. (Note that $1 \le y-x \le N$.) Since $N+1 = R(n,2,3)$, there is a monochromatic triangle $\{x,y,z\}$ (with $x<y<z$) of colour c_i. This means that $y-x, z-y, z-x \in C_i$. But $(y-x)+(z-y) = z-x$; so C_i contains two numbers and their sum.

Thus the theorem is proved, with $f(n) = R(n,2,3) - 1$.

(b) If the set of natural numbers is partitioned into a finite number n of subsets, then there are two numbers x and y such that x, y and $x+y$ belong to the same subset.

Proof. As in the previous proof, we colour the 2-sets of natural numbers with colours $c_1,\ldots,c_n$ by the rule that the 2-set $\{x,y\}$ (with $x<y$) has colour c_i if and only if $y-x$ belongs to the ith subset C_i. By the infinite Ramsey theorem, there exists a monochromatic triangle $\{x,y,z\}$ (with $x<y<z$). Then $y-x, z-y, z-x$ belong to the same subset and $(y-x)+(z-y) = z-x$.

8 A *delta-system* is a family of sets whose pairwise intersections are all equal. (So, for example, a family of pairwise disjoint sets is a delta-system.) Prove the existence of a function f of two variables such that any family $\mathcal{F}$ of at least $f(n,k)$ sets of cardinality n contains k sets forming a delta-system.
State and prove an infinite version of this theorem.

Do you regard this theorem as part of 'Ramsey theory'?

(a) Since there just 2^n subsets of a n-set, as long as the family of k-sets contains more than $1+(2^n-1)(r-1)$ members, for any set A in the family, there must be r sets all having the same intersection with A.

If A_{i-1} and $\mathcal{F}_{i-1}$ have been chosen, let A_i be any member of $\mathcal{F}_{i-1}$, and let $\mathcal{F}_i$ be a subfamily all of whose members have the same intersection with A_i. By the preceding paragraph, we can ensure that $|\mathcal{F}_i| > |\mathcal{F}_{i-1}|/2^k$. So the process can be continued for m steps provided the original family has more than 2^{mk} members.

Now suppose that the series continues for $(k-1)(n+1)+1$ terms. Now $A_i \cap A_j$ depends only on i, not j (if $i<j$); call this intersection B_i. If $i<j<k$, then both A_j and A_k contain B_i, and so $B_j \supseteq B_i$. But $|B_i| \leq n$, so $B_{i+1} \supset B_i$ can hold at most $n+1$ times. By the Pigeonhole Principle, there is a run of at least k consecutive values of i for which B_i remains constant. The corresponding sets A_i for a delta-system with k terms.

(b) An infinite family of n-sets must contain an infinite subfamily which is a delta-system.

Proof. In the above argument, there are infinitely many sets all having the same intersection with A, and so we can choose an infinite sequence $A1, A_2, \dots$ such that all the families $\mathcal{F}_i$ are infinite. Again, the intersection $B_i = A_i \cap A_j$ (for $i<j$) can change at most n times, so it remains constant after some value i_0, and the sets $A_{i_0+1}, A_{i_0+2}, \dots$ form an infinite delta-system.

(c) These results assert that a large structure (here a family of n-sets) must contain a smaller, more homogeneous substructure (a delta-system). This is the general form of a theorem of Ramsey theory.

9 Why are constructive existence proofs more satisfactory than non-constructive ones?

9. This is an 'essay question'. Some points:

- An application (especially one outside mathematics) may require an explicit example of the object whose existence is asserted. For example, an existence theorem for a graph describing the connections on a chip would be of no use to the chip manufacturer unless it has a constructive proof.

- Some schools of philosophy of mathematics reject the law of excluded middle (and hence proof by contradiction), and accept only constructive proofs.
- On the other hand, a non-constructive proof may give a better bound, and may suggest how random search could be ‘directed’ to increase the chance of finding the object in question.