Solutions to Exercises
Chapter 5: The Principle of Inclusion and Exclusion

1 An opinion poll reports that the percentage of voters who would be satisfied with each of three candidates A, B, C for President is 65%, 57%, 58% respectively. Further, 28% would accept A or B, 30% A or C, 27% B or C, and 12% would be content with any of the three. What do you conclude?

By PIE, the percentage of voters who reject all candidates is

$$100 - 65 - 57 - 58 + 28 + 30 + 27 - 12 = -7,$$

so there must be a mistake.

2 Make tables of the two kinds of Stirling numbers for small values of n and k.

1					1			
−1	1				1	1		
2	−3	1			1	3	1	
−6	11	−6	1		1	7	6	1

(The easiest way is to use the recurrence relations of (5.3.2).)

3 Prove directly that $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, and $S(n,n-1) = \binom{n}{2}$. Find a formula for $S(n,n-2)$.

An n-set has only one partition into one part. It has $2^n - 2$ subsets other than the empty set and the entire set; these fall into $2^{n-1} - 1$ complementary pairs, forming all the partitions into two parts.

A partition with $n-1$ parts has one part of size 2 (which can be chosen in $\binom{n}{2}$ ways), and all the others of size 1.

There are two types of partition with $n-2$ parts: either one part of size 3 and the rest of size 1, or two of size 2 and the rest of size 1. So

$$S(n,n-2) = \binom{n}{3} + \binom{n}{2}\binom{n-2}{2}/2 = n(n-1)(n-2)(3n-5)/24.$$

4 Prove that $|s(n,1)| = (n-1)!$ using the recurrence relation, and show directly that the number of cyclic permutations of an n-set is $(n-1)!$.

We have $|s(n+1,1) = n|s(n,1)| + |s(n,0)| = n|s(n,1)|$. Since $|s(1,1)| = 1$, the result follows by induction.

A cyclic permutation is specified by writing the n elements of the set $\{1,\ldots,n\}$ inside a bracket in order (in $n!$ possible ways). However, the representation as a cycle can start at any point, so each permutation has n representations. Thus there are $n!/n = (n-1)!$ cyclic permutations.

5 This exercise outlines a proof that $t^n = \sum_{k=1}^{n} S(n,k)(t)_k$.

(a) Let t be a positive integer, $T = \{1,\ldots,t\}$, and $N = \{1,\ldots,n\}$. The number of functions $f : N \to T$ is t^n. Given such a function f, define an equivalence relation $\equiv$ on N by the rule

$$i \equiv j \quad \text{if and only if} \quad f(i) = f(j).$$

The classes of this equivalence relation can be numbered $C_1,\ldots,C_k$ (say), ordered by the smallest points in the classes. (So C_1 contains 1; C_2 contains the smallest number not in C_1; and so on.) Then the values $f(C_1),\ldots,f(C_k)$ are k distinct elements of T, and so can be chosen in $(t)_k$ ways; the partition can be chosen in $S(n,k)$ ways. Summing over k proves the identity *for the particular value of* t.

(b) Prove that if a polynomial equation $F(t) = G(t)$ is valid for all positive integer values of the argument t, then it is the polynomials F and G are equal.

(a) Follow the outline. The relation $\equiv$ is easily seen to be an equivalence relation, and f induces an injection from the set of its equivalence classes to $\{1,\ldots,t\}$. There are thus $S(n,k)(t)_k$ functions with just k values. (For example, with $t = 3$, there are t functions with $f(1) = f(2) = f(3)$; $t(t-1)$ functions with $f(1) = f(2) \neq f(3)$, and similarly for the other two cases where two values are equal; and $t(t-1)(t-2)$ functions with all values distinct.) So $t^n = \sum_{k=1}^{n} S(n,k)(t)_k$ for this value of t.

(b) let $H(t) = F(t) - G(t)$. If H is not identically zero, and its degree is m, then it can have at most m roots. So, if $H(t) = 0$ for every natural number t, then $H(t)$ is the zero polynomial.

Now apply this with $F(t) = t^n$ and $G(t) = \sum_{k=1}^{n} S(n,k)(t)_k$.

6 For this exercise, recall the Bernoulli numbers $B(n)$ from Exercise 19 of Chapter 4, especially the fact that their e.g.f. is $t/(\exp(t)-1)$. Derive the formula

$$B(n) = \sum_{k=1}^{n} \frac{(-1)^k k! S(n,k)}{(k+1)}$$

for the n^{th} Bernoulli number.

We have

$$G(t) = \frac{t}{\exp(t)-1} = \sum_{n\geq 0} \frac{B(n)t^n}{n!}.$$

On the other hand, we have

$$F(t) = \frac{\log(1+t)}{t} = \sum_{n\geq 0} \frac{(-1)^n t^n}{n+1}.$$

Since $G(t) = F(\exp(t)-1)$, (5.4.2) gives

$$B(n) = \sum_{k=1}^{n} \frac{S(n,k)(-1)^k k!}{k+1}.$$

7 Let (f_n) and (g_n) be sequences, with e.g.f.s $F(t)$ and $G(t)$ respectively. Show the equivalence of the following assertions:

(a) $g_n = \sum_{k=0}^{n} \binom{n}{k} f_k$;

(b) $G(t) = F(t)\exp(t)$.

Assuming (b), we have

$$g_n = n! \sum_{k=0}^{n} \frac{f_k}{k!} \frac{1}{(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} f_k.$$

The converse is proved by reversing the argument.

8 Show that a permutation which is a cycle of length m can be written as a product of $m-1$ transpositions. Deduce that it is an even permutation if and only if its length is odd. Hence show that an arbitrary permutation is even if and only if it has an even number of cycles of even length (with no restriction on cycles of odd length).

$$(a_1\, a_2\, \ldots\, a_m) = (a_1\, a_2)(a_1\, a_3)\ldots(a_1\, a_m),$$

as can be seen by considering the effect of both sides on any point. (On the right, a_1 is mapped to a_2 by the first factor and fixed by the others; for $1 < i < m$, a_i is mapped to a_1 by the $(i-1)$st factor and then to a_{i+1} by the ith, being unchanged by all other factors; and a_m is fixed by all factors but the last, which maps it to a_1. This is exactly the specification of the cycle on the left.)

The parity of the number of transpositions is thus opposite to that of the cycle length.

A permutation has even parity if and only if it contains an even number of cycles of odd parity (even length), with the number of cycle of even parity (odd length) being irrelevant.

9 This exercise outlines the way in which the sign of permutations is normally treated by algebraists. Let $x_1,\ldots,x_n$ be indeterminates, and consider the polynomial

$$F(x_1,\ldots,x_n) = \prod_{i<j}(x_j - x_i).$$

Note that every pair of indeterminates occur together once in a bracket. If π is a permutation, then $F(x_{1\pi},\ldots,x_{n\pi})$ is also the product of all possible differences (but some have had their signs changed). So

$$F(x_{1\pi},\ldots,x_{n\pi}) = \mathrm{sign}(\pi)F(x_1,\ldots,x_n),$$

where $\mathrm{sign}(\pi) = \pm 1$ is the number of pairs $\{i,j\}$ whose order is reversed by π. Prove that

- sign is a homomorphism;
- if τ is a transposition, then $\mathrm{sign}(\tau) = -1$.

For brevity, let F^π denote $F(x_{1\pi},\ldots,x_{n\pi})$. Then the sign function is defined in this exercise by $\mathrm{sign}(\pi) = F^\pi/F$.

(a) Let π,σ be permutations. Let $y_i = x_{i\pi}$ for $i = 1,\ldots,n$. Then $F^\pi = F(y_1,\ldots,y_n)$ and $F^{\pi\sigma} = F(y_{1\sigma},\ldots,y_{n\sigma})$. Thus

$$F^{\pi\sigma}/F^\pi = \mathrm{sign}(\sigma).$$

Multiplying both sides by $F\pi/F = \text{sign}(\pi)$, we have

$$\text{sign}(\pi\sigma) = \text{sign}(\pi)\text{sign}(\sigma),$$

so that sign is a homomorphism, as required.

(b) Let τ be the transposition $(i\ j)$ for $i < j$. Which factors change sign in F^τ? The factor $(x_j - x_i)$ changes. Any factor involving neither x_i nor x_j is unaffected. The factors $(x_k - x_i)$ and $(x_k - x_j)$ are interchanged for $k > j$, and the factors $(x_i - x_k)$ and $(x_j - x_k)$ are interchanged for $k < i$. If $i < k < j$, then $(x_k - x_i)$ maps to the negative of $(x_j - x_k)$, and *vice versa*; these sign changes cancel. So the net sign change is odd, and we have $\text{sign}(\tau) = -1$.

It follows that $\text{sign}(\pi) = (-1)^m$ if π is a product of m transpositions. Moreover, the kernel of sign, the set of permutations with sign $+1$, is a normal subgroup of the symmetric group with index 2, and hence order $n!/2$, for $n \geq 2$.

10 Recall from Section 3.8 that a *preorder* is a reflexive and transitive relation which satisfies trichotomy. Prove that the exponential generating function for the number of preorders on an n-set is $1/(2 - \exp(t))$.

There are $n!$ orders on a set of size n; so the e.g.f. for the number of orders is $\sum_{n \geq 0} n!t^n/n! = 1/(1-t)$.

According to Chapter 3, Exercise 19, a preorder on $\{1, \ldots, n\}$ is specified by a partition on this set (into k parts, say), and an order on the set of parts. So the number of preorders is given by $p_n = \sum_{k=1}^{n} S(n,k)k!$.

By (5.4.2), we have

$$\sum_{n \geq 0} \frac{p_n t^n}{n!} = \frac{1}{1 - (\exp(t) - 1)} = \frac{1}{2 - \exp(t)}.$$

11 (a) Show that the smallest number of transpositions of $\{1, \ldots, n\}$ whose product is an n-cycle is $n - 1$.

(b) Prove that any n-cycle can be expressed in n^{n-2} different ways as a product of $n - 1$ transpositions.

By (5.5.2), the number of cycles of $\pi\tau$ exceeds that of π by at most 1, if τ is a transposition. If π is an n-cycle and is a product of m transpositions $\tau_1, \ldots, \tau_m$, then $\pi\tau_m \ldots \tau_1$ is the identity, with n cycles of length 1; so $m \geq n - 1$.

We use the fact that a graph with n vertices and $n - 1$ edges is connected if and only if it is a tree (see Section 11.2). To each transposition $(i\ j)$ there is a corresponding edge $\{i, j\}$ on the vertex set $\{1, \ldots, n\}$. We claim that the product of $n-1$ transpositions is an n-cycle if and only if the corresponding edges form a tree. One way round, if the graph is not a tree, then there are two points which cannot be connected by a path, and no permutation composed of the transpositions can carry one to the other.

For the converse, let $\tau_1, \ldots, \tau_{n-1}$ correspond to the edges $e_1, \ldots, e_{n-1}$ of a tree. We show by induction on m that the graph G_m with edge set $\{e_1, \ldots, e_m\}$ has $n-m+1$ connected components, each of which is a tree, and $\tau_1 \cdots \tau_m$ is a product of $n-m+1$ distinct cycles, one on each component of G_m. This is clear for $m=0$ (or 1). Assume that it holds for m. Then e_{m+1} joins vertices in different components of G_m, so composing with τ_{m+1} stitches two of the cycles of $\tau_1 \cdots \tau_m$ together, completing the inductive step.

By Cayley's Theorem, there are n^{n-2} trees on $\{1, \ldots, n\}$; each has $n-1$ edges, which can be ordered arbitrarily, so there are $n^{n-2}(n-1)!$ products of n transpositions which form a single cycle. By symmetry, each of the $(n-1)!$ cycles (see Exercise 4) occurs equally often, necessarily n^{n-2} times.