# The Erdős–Ko–Rado theorem

What is the size of the largest intersecting family of $k$-subsets of an $n$-set? There is an intersecting family of size $\binom{n-1}{k-1}$ consisting of all $k$-sets containing a fixed element. If $n < 2k$, we can do better: any two $k$-sets intersect, so the maximum size is $\binom{n}{k}$.

The Erdős–Ko–Rado theorem answers the question by showing that, if $n \geq 2k$, the examples constructed above are optimal: that is, no intersecting family is larger. Moreover, if $n > 2k$, they are the only examples which meet the bound.

This is one of the landmark results of extremal set theory. It was published by the authors, Paul Erdős, Chao Ko, and Richard Rado in 1961, though the proof had been found over two decades earlier.

We begin with a lemma of independent interest. A *clique* in a graph is a set of vertices, any two of which are joined. A graph $G$ is *vertex-transitive* if, for any two vertices $x$ and $y$ of $G$, there is an automorphism $g$ of $G$ with $x^g = y$.

**Lemma 1** *Let $G$ be a vertex-transitive graph with vertex set $X$. Let $Y$ be a subset of $X$ with the property that any clique in $Y$ has size at most $|Y|/m$. Then any clique in the graph has size at most $|X|/m$. A clique $C$ meeting the bound satisfies $|C^g \cap Y| = |Y|/m$ for all automorphisms $g$ of $G$.*

**Proof** Let $N$ be the order of the automorphism group of $G$. By vertex transitivity, for any two vertices $x$ and $y$, the number of automorphisms $g$ satisfying $x^g = y$ is $N/|X|$.

Count pairs $x, g$, where $x$ is a vertex in $C$ and $g$ an automorphism such that $x^g \in Y$. There are $|C|$ choices for $x$, and $N/|X|$ choices for $g$ satisfying $x^g = y$ for each of the $|Y|$ possible $y \in Y$. So the number of pairs is $|C| \cdot |Y| \cdot N/|X|$.

On the other hand, for each of the $N$ automorphisms $g$, there are at most $|Y|/m$ choices of $x \in X$ (in the inverse image of $X^g \cap Y$).

So $|C| \cdot |Y| \cdot N/|X| \leq N \cdot |Y|/m$, whence $|C| \leq |X|/m$ as required.

The characterisation of equality is clear from the proof.

A *coclique* in a graph is a set of vertices with no pair joined. Since a clique and a coclique intersect in at most one point, the lemma shows that, in a vertex-transitive graph $G$ with vertex set $X$, any clique $C$ and coclique $D$ satisfy $|C| \cdot |D| \leq |X|$. (Take $Y = D$ and $m = |D|$.) Simple examples show that this is false in arbitrary regular graphs.

**Theorem 2** *Suppose that $n \geq 2k$. Then an intersecting family of $k$-subsets of an $n$-set has cardinality at most $\binom{n-1}{k-1}$.*

**Proof** Consider the graph $G$ whose vertices are the $k$-subsets of an $n$-set, adjacent of their intersection is non-empty. The graph is vertex-transitive (since any permutation of the $n$-set is an automorphism) and has $\binom{n}{k}$ vertices. We have to show that the size of a clique is at most $\binom{n-1}{k-1} = \binom{n}{k}/(n/k)$. So, by the lemma, it suffices to find a family $Y$ of $k$-sets with $|Y| = n$ such that any intersecting subfamily of $Y$ has size at most $k$.

Take the $n$-set to be the integers mod $n$, and consider the set of all $n$ intervals of length $k$ (regarding $X$ as cyclically ordered). It is clear that $k$ intervals with consecutive starting points form a clique. We must show that there is no larger clique. Any such clique would have to contain three intervals covering the entire circle, say $[0,k-1]$, $[k-x,2k-x-1]$ and $[2k-x-y,3k-x-y-1]$, with $3k-x-y-1 \geq n$. The size of such a clique is at most $x+y+(3k-x-y-n) = 3k-n \leq k$, since $n \geq 2k$. So the claim is proved.

We can use the characterisation of equality in the lemma to find all the extremal families, if $n > 2k$. (If $n = 2k$, then any choice of one of each complementary pair of $k$-sets gives an intersecting family of size $\frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}$.)

**Theorem 3** *If $n > 2k$, then any intersecting family of $k$-subsets of an $n$-set of cardinality $\binom{n-1}{k-1}$ consists of all the $k$-sets containing some fixed point.*

**Proof** Let $\mathcal{F}$ be an intersecting family of cardinality $\binom{n-1}{k-1}$. We start with two observations.

First, suppose that there are two points $x$ and $y$ such that every $k$-set containing $x$ but not $y$ belongs to $\mathcal{F}$. Then $\mathcal{F}$ consists of all the $k$-sets containing $x$. For, given a $k$-set $K$ not containing $x$, there is a $k$-set $L$ containing $x$ but not $y$ disjoint from $K$ (since there are at least $2k$ points different from $y$). Thus $K \notin \mathcal{F}$. So every set in $\mathcal{F}$ conntains $x$, and by considering the cardinality we must have every such set. So, if $\mathcal{F}$ is a counterexample, then for every $x$ and $y$ there is a $k$-set not in $\mathcal{F}$ containing $x$ but not $y$.

Second, there are two $k$-sets $K, K'$ intersecting in $k-1$ points, such that $K$ is in $\mathcal{F}$ and $K'$ is not. For, if not, then for each $K \in \mathcal{F}$, every $k$-set meeting $K$ in $k-1$ points is also in $\mathcal{F}$; by induction, every $k$-set is in $\mathcal{F}$, which is impossible.

Choose $K, K'$ as above, and label the points in $K \setminus K'$ and $K' \setminus K$ as $0$ and $k$ respectively. Assuming that $\mathcal{F}$ is a counterexample to the theorem, choose $K'' \notin \mathcal{F}$ with $0 \in K''$ and $k \notin K''$. Let $K \cap K'' = \{0, \ldots, l-1\}$ with $l < k$; then number the remaining points of $K$ as $l, \ldots, k-1$, and the remaining points of $K''$ as $n-k+$

$l, \ldots, n-k+1$. Number the remaining points with the remaining elements of the integers mod $n$.

By the lemma, $\mathcal{F}$ contains $k$ intersecting intervals on the cycle, and by the proof of the theorem, these intervals must be consecutive. But we have chosen the sets so that this is impossible, since the interval $[0, k-1]$ is in $\mathcal{F}$ while the intervals $[1, k]$ and $[n-k+l, l-1]$ are not.

This contradiction proves the theorem.