Graph homomorphisms

Peter J. Cameron

Combinatorics Study Group Notes, September 2006

Abstract

This is a brief introduction to graph homomorphisms, hopefully a prelude to a study of the paper [1].

1 Homomorphisms

A *homomorphism* from a graph G to a graph H is a map from VG to VH which takes edges to edges. (It may map a nonedge to a single vertex, a nonedge, or an edge.)

Homomorphisms are a generalisation of graph colourings. A homomorphism from the graph G to the complete graph K_r (with vertices numbered $1, 2, \ldots, r$) is exactly the same as an r-colouring of G (where the colour of a vertex is its image under the homomorphism), since adjacent vertices map to distinct vertices of the complete graph. Thus homomorphisms generalise colourings.

Example: Scheduling How to schedule the exams in the smallest number of periods? Two exams taken by the same student cannot be scheduled at the same time. So make a graph G whose vertices are the exams, two vertices joined by an edge if some student is taking both exams. Then an exam schedule in k periods exists if and only if the graph can be coloured with k colours, that is, there is a homomorphism from G to the complete graph K_k.

Now suppose that a student is not permitted to take exams in consecutive periods. Let H_k be the complete graph on k vertices with the $k-1$ edges of a path removed. Then the exams can be scheduled in k periods if and only if there is a homomorphism from G to H_k.

We see that there is a close connection between homomorphism and constraint satisfaction problems.

We say $G \to H$ if there is a homomorphism from G to H, and $G \equiv H$ if $G \to H$ and $H \to G$. Then $\to$ is a preorder (it is reflexive because the identity map is a homomorphism, and transitive because the composition of homomorphisms is a homomorphism) and $\equiv$ is the derived equivalence relation (called *homomorphism equivalence*). The equivalence classes are partially ordered by $\to$; this is the *homomorphism order*. Sometimes we abuse notation by referring to the order on individual graphs.

Since homomorphisms map edges to edges, we see that a homomorphic image of a connected graph must be connected.

Let $\omega(G)$ and $\chi(G)$ denote the clique number and chromatic number of the graph G. Now the clique number of G is the largest value of k for which $K_k \to G$, since the image of a complete graph under a homomorphism is a complete graph of the same size. Also, a graph is k-colourable if and only if it has a homomorphism to the complete graph K_k; so $\chi(G)$ is the smallest k for which this holds.

Proposition 1.1 *If $G \to H$, then $\omega(G) \leq \omega(H)$ and $\chi(G) \leq \chi(H)$.*

Proof If $\phi : G \to H$ is a homomorphism, then composing with homorphisms from or to K_k we see that

- $K_k \to G$ implies $K_k \to H$;
- $H \to K_k$ implies $G \to K_k$.

Corollary 1.2 *If $G \equiv H$, then $\omega(G) = \omega(H)$ and $\chi(G) = \chi(H)$.*

2 Cores

A graph G is a *core* if it has the minimum number of vertices of any graph in its homomorphism equivalence class.

Proposition 2.1 *If G is a core, then every endomorphism of G is an automorphism.*

Proof Let ϕ be an endomorphism of G, and let H be the induced subgraph of G on $(VG)\phi$. Then ϕ induces a homomorphism $G \to H$, and inclusion a homomorphism $H \to G$. So $G \equiv H$. If ϕ is not onto, then G is not of minimum size in its equivalence class.

We will see shortly that the converse is also true.

Proposition 2.2 *Every homomorphism equivalence class contains a unique core (up to isomorphism).*

Proof It's clear that any homomorphism equivalence class contains a core. Moreover, two equivalent cores are isomorphic. For if G and G' are cores and $G \equiv G'$, then there are homomorphisms $\phi : G \to G'$ and $\phi' : G' \to G$ such that $\phi\phi'$ and $\phi'\phi$ are endomorphisms of G and G'; hence they are isomorphisms. Now ϕ is a bijective endomorphism, so cannot decrease the number of edges, and similarly for ϕ'; so ϕ and ϕ' are isomorphisms.

In particular, the homomorphism order on equivalence classes of graphs is the same as the homomorphism order on isomorphism classes of cores.

We say that G is a *core* of G' if it is an induced subgraph of G' which is a core.

Proposition 2.3 *Any graph has a unique core (up to isomorphism).*

Proof Take an arbitrary graph H, and let G be the core of its equivalence class. There is a homomorphism $\phi : G \to H$; the induced subgraph G' on $(VG)\phi$ satisfies $G \to G' \to H$, and $|VG| = |VG'|$, so G' is another core, and G' is isomorphic to G.

Now we have the promised converse:

Proposition 2.4 *A graph G is a core if and only if every endomorphism of G is an automorphism.*

Proof We saw the forward implication already. Conversely, suppose that every endomorphism of G is an automorphism, and let H be a core of G. Then by definition there is a homomorphism from G to H; followed by the embedding of H in G, this is an endomorphism of G, and so an automorphism. So $G = H$.

Proposition 2.5 *If the clique number and chromatic number of G are both equal to k, then the core of G is K_k.*

Proof By our characterisations of clique and chromatic number in the first section, we see that $K_k \to G$ and $G \to K_k$, so $G \equiv K_k$. Now, if G is equivalent to a graph H on fewer than k vertices, then we would have $\chi(G) \leq \chi(H) < k$, contrary to hypothesis.

We see that bipartite graphs with at least one edge form a homomorphism equivalence class with core K_2.

Proposition 2.6 *If G is vertex-transitive then so is its core.*

Proof Let H be the core of G and choose homomorphisms $\phi : H \to G$ and $\psi : G \to H$. Then ϕ is an embedding of H as induced subgraph of G, and ϕ and ψ are isomorphisms between H and $G|_{(VH)\phi}$, so that $\phi\psi$ is an automorphism of H. Now, for any automorphism α of G, $\phi\alpha\psi$ is an automorphism of H. We can choose α to map any vertex in $(VH)\phi$ to any other; so the automorphism group of H is vertex-transitive.

Proposition 2.7 *The core of an edge-transitive graph is edge-transitive; the same is true for arc-transitivity and for (ordered or unordered) clique-transitivity.*

Proof As in the preceding result.

Question 1 *What about other forms of transitivity?*

Question 2 *Is there a direct way to recognise vertex-transitive cores, or a sufficient condition in terms of the automorphism group for a graph to be a core?*

A core need not be connected. For example, let G_3 be the Grötzsch graph on 11 vertices, with clique number 2 and chromatic number 4. It is not hard to see that G_3 is a core. (In fact, there is no need to show this; simply replace G_3 by its core in the construction below.)

Now let G be the disjoint union of K_3 and G_3. There is no homomorphism in either direction between K_3 and G_3, because of the relative sizes of their clique and chromatic numbers. So every endomorphism of the disjoint union maps the components to themselves; since the components are cores, such an endomorphism is an automorphism, and so G is a core. This example is typical:

Proposition 2.8 *The connected components of a core are cores of classes forming an antichain in the homomorphism order. Conversely, if* $\{G_1, \ldots, G_r\}$ *is an antichain of cores, then the disjoint union of these graphs is a core.*

Proof Let $G_1, \ldots, G_r$ be the connected components. If $G_i \to G_j$ for $i \neq j$, then combining this with the identity on the other components gives a homomorphism which is not an automorphism, which is not possible. So $\{G_1, \ldots, G_r\}$ is an antichain. If any of these graphs G_i were equivalent to a smaller graph G_i', then the same argument would show that G would be equivalent to a smaller graph. So $G_1, \ldots, G_r$ are cores.

The converse is proved in the same way as in the example.

Corollary 2.9 *The core of a vertex-transitive graph is connected.*

Proof Note that there are two quite different ways to see this. We can use the fact that the core is vertex-transitive, and the preceding proposition then shows that it is connected; or observe that a disconnected vertex-transitive graph is equivalent to one of its connected components, whose core is obviously connected.

Example There is a homomorphism from C_{n+2} to C_n for any n: simply "fold" two consecutive edges of the longer cycle onto a single edge of the shorter. Now each odd cycle is a core, since it is not bipartite but all its proper subgraphs are. Also, for odd n, there is no homomorphism from C_n to C_{n+2}. So the odd cycles form an infinite descending chain between K_2 and K_3 in the homomorphism order. (As we saw, all even cycles are equivalent to an edge.)

3 Counting homomorphisms

We want to look at the question: How many homomorphisms are there from a graph F to a graph G? Let $\mathrm{Hom}(F, G)$ denote the set of homomorphisms from F to G. Now $|\mathrm{Hom}(K_r, G)|$ is $r!$ times the number of r-cliques in G, and so is non-zero if and only if $\omega(G) \geq r$; amd $|\mathrm{Hom}(G, K_r)|$ is the number of r-colourings of G, and is non-zero if and only if $r \geq \chi(G)$.

The homomorphisms from K_r to G, or from G to K_r, are thus counted by the clique and chromatic polynomials of G. These are not enough to distinguish graphs. For example, any n-vertex tree has $n-1$ edges and no larger cliques, and has chromatic polynomial $x(x-1)^{n-1}$.

In 1967, Lovász [2] proved the following theorem.

Theorem 3.1 (Lovász) *Let G and G' be finite graphs, and suppose that $|\operatorname{Hom}(F,G)| = |\operatorname{Hom}(F,G')|$ for all finite graphs F. Then $G \cong G'$.*

Proof The first step is to show that $|\operatorname{Mon}(F,G)| = |\operatorname{Mon}(F,G')|$ for all finite graphs F, where $\operatorname{Mon}(F,G)$ is the set of monomorphisms from F to G. Each homomorphism from F to G can be uniquely decomposed into a *canonical epimorphism* from F to a graph H, followed by a monomorphism from H to G. (Call two vertices of F equivalent if their images under the given monomorphism are equivalent. Then the vertices of H are the equivalence classes, numbered according to their smallest member in the numbering of the vertices of F; two vertices of H are adjacent if some representatives in F are adjacent.)

By induction we may suppose that $|\operatorname{Mon}(H,G)| = |\operatorname{Mon}(H,G')|$ for any graph H with fewer vertices than F. Now $|\operatorname{Hom}(F,G)|$ is the sum of terms $|\operatorname{Mon}(H,G)|$ over all graphs H which are images of canonical epimorphisms from F. By induction all these terms agree except possibly when $H = F$, in which case the remaining terms $|\operatorname{Mon}(F,G)|$ must also agree.

Hence $|\operatorname{Mon}(G,G')| = |\operatorname{Mon}(G,G)|$, so there exists a monomorphism from G to G'; and similarly a monomorphism from G' to G. Their compositions either way round are automorphisms of G and G'. So they are isomorphisms.

The theorem says that a graph G is determined by the sequence

$$(|\operatorname{Hom}(F_1,G)|, |\operatorname{Hom}(F_2,G)|, \ldots),$$

where $F_1, F_2, \ldots$ is a list of all finite graphs. The authors of [1] use this to define a metric on the space of finite graphs, so that two graphs are close together if the representing sequences are close on some initial segment. They proceed to develop theoretical and practical consequences of this definition, with spin-offs in complexity, statistical mechanics, and so on. We will hopefully learn more about this.

References

[1] Ch. Borgs, J. Chayes, L. Lovász, V. T. Sós and K. Vesztergombi, Counting graph homomorphisms, pp.315–371 in *Topics in Discrete Mathematics* (Algorithms and Complexity **26**), ed. M. Klazar *et al.*, Springer, 2006.

[2] L. Lovász, Operations with structures, *Acta Math. Hungar.* **18** (1967), 321–328.