Graph homomorphisms Ⅲ: Models

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Abstract

Following the last talk on graph homomorphisms, we continue to discuss some examples of graph homomorphisms. But this time we will focus on some models, that is, the homomorphism $G \to H$ for the graph H with fixed weights. The main reference is Section 1 in [2].

1 Introduction

The basic setting is the same as that of last talk: $G = (V(G), E(G))$ is a simple graph unless stated otherwise; $\phi : G \to H$ is a homomorphism from G to H and $hom(G, H)$ is the number of (weighted) homomorphisms from G to H.

But this time we will focus on the weights on H as well as H itself. More precisely, a model for G is a weighted graph (H, ω, Ω), where ω maps each vertex/edge to an element of the communative ring Ω. Because H can be assumed to be a complete graph on q points with a loop on each vertex and in this talk each vertex of H will be associated with weight 1. We denote $V(H)$ by S and now w is a symmetric function:

$$w : \; S \times S \to \Omega$$

Elements of S are called spins or *colors* and the function w is known as the *Boltzmann weight.*

We call $M = (S, \omega, \Omega)$ is a *spin model* for G. In this setting each map from G to H is a homomorphim, and also called a state of G. $hom(G, H)$ is the paritition function:

$$Z_G^M = \sum_{\sigma : G \to S} \prod_{uv \in E(G)} w(\sigma(u), \sigma(v)) \in \Omega$$

of G. Where M depends on parameters such as T (temperature) and Z_G^M is a "function" on T.

$G \to Z_G^M$ is a *graph function*. On the other hand, we can fix S and Ω and each graph G provides an Ω-valued function $w \to Z_G^{(S,w,\Omega)}$.

2 Resonant models

In this section we assume $S = \{1, \cdots, q\}$, $\Omega = \mathbb{C}$ and ω has the form $w(s,t) = \alpha$ when $s = t$ and $w(s,t) = \beta$ otherwise. This model is called resonant models in [1]. And the Potts model is the specification when $\alpha\beta = 1$, and Ising model is a specification of Potts model when $p = 2$.

Let

$$t_{\alpha,\beta}(s,t) = \alpha\delta_{s,t} + \beta(1 - \delta_{s,t}).$$

Then we have $w(\sigma(u), \sigma(v)) = t_{\alpha,\beta}(\sigma(u), \sigma(v))$.

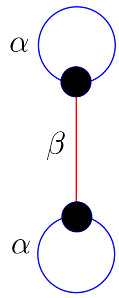

Figure 1: Resonant model

Example 1. *Rudimentary example*
Let $\alpha = \beta$. Then $Z_G = q^{|V(G)|}\alpha^{|E(G)|}$. In particular, from $q = 2$ and $\alpha = 1$ we can work out $|V(G)|$ and $|E(G)|$ from $q = 1$ and $\alpha = 2$.

Example 2. Now let $\beta = 0$ and $q \geq 2$. Given $\phi : G \to H$, we have $t_{\alpha,\beta}(\phi(u), \phi(v)) = \alpha\delta_{(\phi(u),\phi(v))}$ and the weight function is:

$$hom_\phi^M(G,H) = \prod_{uv \in E(G)} w(\phi(u), \phi(v)) = \prod_{uv \in E(G)} \alpha\delta_{(\phi(u),\phi(v))}$$

which is zero if $\phi(u) \neq \phi(v)$ for some $uv \in E(G)$. Consequently, the only state which make a non-zero contribution to the partition function are those which map any edge to a loop. In other words, such ϕ must be constant on each component of G. If G has $k(G)$ components, then there are $p^{k(G)}$ such states (maps), and each one contributes a weight $\alpha^{|E(G)|}$. Thus

$$Z_G^M = p^{k(G)} \cdot \alpha^{|E(G)|}.$$

2.1 Ising model

Example 3. *Ising and Potts model*
Consider $S = \{1, \cdots, q\}$ as above, a ring Ω and two elements $\alpha, \beta \in \Omega$. These date define a mode $M_{\alpha,\beta}$ for which $w(s,t) = \alpha$ if $s = t$ and $w(s,t) = \beta$ otherwise. The few models described so far are examples of these $M_{\alpha,\beta}$'s. In the present discussion, the *potts model* is the particular case in which $\Omega = \mathbb{C}, \alpha \neq 0$, and $\beta = \alpha^{-1}$; the *zero-field Ising model* is the Potts model with moreover $q = 2$.

Physical motivation Let a graph G represent a crystal. Assume that, in a state $\sigma : V(G) \to S$, an edge $uv \in E(G)$ has some energy $-L$ if $\sigma(u) = \sigma(v)$ and $+L$ otherwise, for some positive constant L. In the state σ, the energy of the whole crystal is

$$H(G, \sigma) = -L \sum_{uv \in E(G)} \varepsilon(\sigma(u), \sigma(v)),$$

where $\varepsilon(s,t) = +1$ if $s = t$ and -1 if $s \neq t$. In classical statistical physics, the crystal G will be in the state σ, at temperature T, with a probability proportional to $e^{1/kT} H(G, \sigma)$, where k denotes the Boltzmann constant; namely with a probability proportional to

$$\exp\left(\frac{L}{kT} \sum_{uv \in E(G)} \varepsilon(\sigma(u), \sigma(v))\right) = \prod_{uv \in E(G)} \exp\left(\frac{L}{kT} \varepsilon(\sigma(u), \sigma(v))\right)$$

If we set

$$\alpha = e^{\frac{L}{kT}}$$

$$w(s,t) = \begin{cases} \alpha & if s = t \\ \alpha^{-1} & if s \neq t \end{cases}$$

The probability attached to σ is precisely

$$\frac{\exp\left(-(1/kT) H(G, \sigma)\right)}{\sum_\tau \exp\left(-(1/kT) H(G, \tau)\right)} = \frac{1}{Z_G^M} \prod_{uv \in E(G)} w(\sigma(u), \sigma(v)).$$

This shows one motivation for computing Z_G^M.

2.2 Eulerian Model

Example 4. *Eulerian Model* The Eulerian Model is $\alpha = 1$ and $\beta = -1$. Then we have $Z_G^M = 2^{|V(G)|}$ if G is an Eulerian graph and 0 otherwise.

Proof 1 For each vertex v of graph G, let $d(v)$ denote the degree of x (each loop at x counting for two). If $w(s,t) = f(s)f(t)$ for some function $f : S \to \Omega$. Then

$$Z_G = \sum_{\sigma} \prod_{uv \in E} f(\sigma(u))f(\sigma(v)) = \sum_{\sigma} \prod_{v \in V(G)} f(\sigma(v))^{d(v)} = \prod_{v \in V(G)} \sum_{s \in S} f(s)^{d(v)}.$$

For Eulerian model, $S = \{0,1\}$ and $\omega(s,t) = f(s)f(t)$ for $f(s) = (-1)^s$. Therefore we have:

$$Z_G = \prod_{v \in V(G)} (1^{d(v)} + (-1)^{d(v)}) = \begin{cases} 2^{|V(G)|} & \text{if } d(x) \text{ is even } \forall x \in V(G) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $Z_G = 2^{|V(G)|}$ if G is an *Eulerian graph*, and $Z_G = 0$ otherwise.

Note: A demonstration $(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) = \sum_\sigma x_{\sigma(x)} y_{\sigma(y)} z_{\sigma(z)}$ where $\sigma : \{x, y, z\} \to \{1, 2\}$. $\square$

Proof 2: Firstly we need the following lemma.

Lemma 2.1. *For any simple graph G, the size of any cut set is even if all vertices have even degree, otherwise the number of cut sets with even size equal to that with odd size.*

Proof. Let $V(G) = V_1 \cup V_2$ be a partition of $V(G)$ into two disjoint subsets. Denote the number of edges in $G|V_1$ by $e(V_1)$, then the cut size is given by:

$$t(V_1, V_2) = (\sum_{v \in V_1} d_v) - 2 \times e(V_1).$$

From which we know the first conclusion holds and the parity of the cut size is the same as the parity of the vertices having odd degree in any set of the partition. Therefore, we can assume all the vertices in G have the odd degree and $V(G) = 2k$ from hand-shaking lemma. Now we can prove the second conclusion by the induction on k. $k = 1$ is easy as there are just two possible unordered partition $\emptyset \cup \{1, 2\}$ and $\{1\} \cup \{2\}$. For any partition of $[2k] = \{A\} \cup \{B\}$, we can form four partitions of $[2k + 2]$: $\{A\} \cup \{B, 2k+1, 2k+2\}$, $\{A, 2k+1, 2k+2\} \cup \{B\}$, $\{A, 2k+1\} \cup \{B, 2k+2\}$, $\{A, 2k+2\} \cup \{B, 2k+1\}$, where the first two do not change the parity of the parity of the origin partition while the remained two change it. Therefore we finish the induction step as each partition of $[2k+2]$ must come from this way. $\square$

For each state σ of G, $V(G) = \sigma^{-1}(a) \cup \sigma^{-1}(b)$ gives an ordered partition of $V(G)$ and we denote the cut size of this partition by $t(\sigma)$. Thus we have:

$$\begin{aligned} Z_G^E &= \sum_{\sigma} \prod_{uv \in E(G)} \omega(\sigma(u), \sigma(v)) \\ &= \sum_{\sigma} \prod_{\substack{uv \in E(G) \\ \sigma(u) \neq \sigma(v)}} (-1) \\ &= \sum_{\sigma} (-1)^{t_\sigma} \\ &= 2 \times \sum_{f \in \Gamma} (-1)^{t(f)} \end{aligned}$$

where Γ denotes the all possible unordered partition and $t(f)$ is the cut size of partition f. Now the conclusion holds follow the lemma. $\square$

Note: we have the following correspondings:

$$\text{maps} \overset{1:1}{\to} \text{Ordered partitions} \overset{2:1}{\to} \text{Unordered partitions} \overset{k:1}{\longrightarrow} \text{cut sets}$$

A variation, which is a particular case of Potts' model: $S = \mathbb{Z}_q$ and $f(s) = \exp 2i\pi s/q$. Similarly $Z_G = q^{|V(G)|}$ if $d(x) \equiv 0 (\bmod q)$ for each $x \in V(G)$ and $Z_G = 0$ otherwise.

Another variation is the non-where zero flow model, which was presented on last talk's note.

2.3 Maximal Cut model

Example 5. (Maximum cut) Let H denote the looped complete graph on two nodes, weighted as follows: the non-loop edge has eight 2; all other edges and nodes have weight 1. Then for every simple graph G with n nodes,

$$\log_2 hom(G, H) - n \leq \text{MaxCut}(G) \leq \log_2 hom(G, H).$$

where $\text{MaxCut}(G)$ denotes the size of the maximum cut in G. So unless G is very sparse, $\log_2 hom(G, H)$ is a good approximation of the maximum cut in G.

Proof. Choose a cut of maximal size, denote by t_{max}, and a homomorphism ϕ_{max} (not unique) corresponding to this cut.

From the weight on H, we know:

$$hom(G,H) = \sum_{\phi:G\to H} hom_\phi(G,H) = \sum_{\phi:G\to H} 2^{t_\phi}$$

where t_ϕ is the cut size corresponding to the partition given by ϕ.

Then we have

$$2^{t_{\max}} = hom_{\phi_{max}}(G,H) \leq hom(G,H),$$

which means

$$\mathrm{MaxCut}(G) \leq \log_2 hom(G,H).$$

On the other hand,

$$hom(G,H) \leq 2^n hom_{\phi_{max}}(G,H) = 2^{n+t_{max}}$$

as there are total 2^n homomorphisms from G to H, which implies:

$$\log_2 hom(G,H) - n \leq \mathrm{MaxCut}(G).$$

□

3 Expanding

Now we are ready to investigate the change of Z_G^M w.r.t the change of models. More precisely, we have two models $M = (\alpha,\beta)$ and $M' = (\alpha',\beta')$ on H with (α,β) transformed from (α',β') by $\alpha = c\cdot\alpha' + d$ and $\beta = c\cdot\beta' + d$ where

$$c = \frac{\beta-\alpha}{\beta'-\alpha'} \qquad d = \frac{\beta'\alpha - \beta\alpha'}{\beta'-\alpha'}.$$

Then we have

Theorem 3.1.

$$\widetilde{Z_G} = d^{|E(G)|}\sum_{A\subset E}(\frac{c}{d})^{|A|}\widetilde{Z'_X}$$

where X denote the graph whose set of edges is A and whose set of vertices consists of the $v \in V(G)$ incident with some edge in A and the summation runs over all subsets of $E = E(G)$ (including the empty set and E itself).

The proof of this theorem is rather technical and we will put it in the appendix. In this section we will investigate how to get the partition functions of complicated models from that of simple ones via Theorem 3.1. For simplicity, we assume $p = 2$ unless stated otherwise. The following three simple models are used as our prototypes.

- Euler model $E : \alpha' = 1, \beta' = -1$

$$\widetilde{Z_G^E} = 1 \text{ if } G \text{ is Eulerian, and 0 otherwise.}$$

- Coloring mode $C : \alpha' = 0, \beta' = 1$

$$Z_G^C = hom(G, K_2) \text{ if } G \text{ is bipartite, and 0 otherwise.}$$

- Local constant model $L : \alpha' = 1, \beta' = 0$

$$Z_G^L = p^{K(G)}$$

Note: for generalized local constant model where $\alpha' \neq 0$, we have $Z_G^L = p^{K(G)} \cdot \alpha'^{|E(G)|}$.

3.1 Expanding from Euler model

Corollary 3.2. *We can obtain the partition function of the Ising model from that of the Eulerian Model.*

Proof. Now we have $\alpha = \varepsilon, \beta = \varepsilon^{-1}$ and $\alpha' = 1, \beta' = -1$ where $\varepsilon = e^{L/kT}$. By solving the following equation

$$\begin{aligned} \varepsilon &= c + d \\ \varepsilon^{-1} &= -c + d \end{aligned}$$

we know

$$\begin{aligned} c &= \tfrac{1}{2}(\varepsilon - \varepsilon) &= \sinh \frac{L}{kT} \\ d &= \tfrac{1}{2}(\varepsilon + \varepsilon) &= \cosh \frac{L}{kT} \end{aligned}$$

Thus from Theorem 3.1 we can get:

$$\widetilde{Z_G^M} = (\cosh \frac{L}{kT})^p \sum_{A \subset E} (\tanh \frac{L}{kT})^{|A|} \widetilde{Z_X^{M'}}$$

Since M' is the Eulerial model, whose normalized partition function is 1 if A is an Eulerian graph and 0 otherwise, the following relations holds.

$$\widetilde{Z_G^M} = (\cos \frac{L}{kT})^p \sum_i N(i)(\tanh \frac{L}{kT})^i$$

where $N(i)$ is the number of edge-subgraphs of G which are Eulerian and have i edges. This relation can be regarded as a series expansion for the partition function of the Ising model. If the temperature T is large, then $\tanh L/kT$ is small, and the series is cosequently known as a "high-temperature expansion" for the Ising model.

□

Note: The set of Eulerian subgraphs is small compared with the set of all subgraphs.

For general model $M = (\alpha, \beta)$, we have the following shifting function:

$$c = \frac{1}{2}(\alpha - \beta) \qquad d = \frac{1}{2}(\alpha + \beta).$$

From Theorem 3.1 we have:

Lemma 3.3.

$$\begin{aligned} \widetilde{Z_G^M} &= (\frac{1}{2}(\alpha+\beta))^2 \sum_{A \subset E} (\frac{\alpha-\beta}{\alpha+\beta})^{|A|} \widetilde{Z_X^{M'}} \\ &= (\frac{1}{2}(\alpha+\beta))^2 \sum_i N(i) (\frac{\alpha-\beta}{\alpha+\beta})^i \end{aligned}$$

Corollary 3.4.

$$Z_G^C = \frac{1}{4} \sum_{A \subset E} (-1)^{|A|} \widetilde{Z_X^E} = \frac{1}{4} \sum_i (-1)^i N(i)$$

$$Z_G^L = \frac{1}{4} \sum_{A \subset E} \widetilde{Z_X^E} = \frac{1}{4} \sum_i N(i)$$

3.2 Expanding from Local constant model

Lemma 3.5. *Let $\beta' = 0$ and $\alpha' \neq 0$, then we can shift it to get an arbitrary model $M = (\alpha, \beta)$.*

Proof. In this case we have $c = \frac{\alpha-\beta}{\alpha'}$ and $d = \beta$. From the theorem we get:

$$\begin{aligned}
\widetilde{Z_G^M} &= \beta^{|E|} \sum_{A \subset E} \left(\frac{\alpha - \beta}{\alpha' \beta} \right)^{|A|} \frac{p^{|k(X)|} \alpha'^{|A|}}{p^{|V(X)|}} \\
&= \beta^{|E|} \sum_{A \subset E} \left(\frac{\alpha}{\beta} - 1 \right)^{|A|} p^{k(X) - |V(X)|} \\
&= \beta^{|E|} \sum_{A \subset E} \left(\frac{\alpha}{\beta} - 1 \right)^{|A|} \left(\frac{1}{p} \right)^{r(X)} \heartsuit
\end{aligned}$$

where $r(X) = |V(X)| - k(X)$ is called the *rank* of X; it is the rank of the incidence matrix of X. □

Remark: The expression of ♡ is an example of the *rank polynomial* of a graph. The fact that the partition function of a resonant model can be expressed in this way has two important implications.

1: The Z depends essentially only on the ration α/β. (projective space rather than the whole plane: consider the point on the circles?)

2: In resonate case, Z is depends only on the number p of configurations available to each particle, not on any algebraic structure we might have imposed on the set S of such configuration. But in some cases it's useful to be able to impose an algebraic structure, even though it does not influence the final result.

Example 6. *Proper colorings* Let $\alpha = 0$, $\beta = 1$. In this case, it's usual to treat p as a variable. The number of poper coloring is given by:

$$Z_G = p^{|V(G)|} \sum_i c_i(G) \left(\frac{1}{p} \right)^i$$

where the coefficients are:

$$c_i(G) = \sum (-1)^{|A|}$$

the sum being taken over all edge-subgraphs $< A >$ of G whose rank is i.

From this expression is easy to see that Z is a polynomial in p, of degree $|V(G)|$. The dominant coefficients are those for small values of i, and these depend only on 'small' subgraphs. For instance, the only edge-subgraph of rank 0 is the null graph, and the only ones of rank 1 are the single edges, so

$$c_0(G) = 1, \qquad c_1(G) = -|E(G)|.$$

Corollary 3.6. *For Euler model, we have* $\alpha = 1$ *and* $\beta = -1$. *Thus we have:*

$$Z_G^E = (-1)^{|E(G)|} p^{|V(G)|} \sum_i f_i(G) \left(\frac{1}{p}\right)^i$$

where the coefficients are:

$$f_i(G) = \sum (-2)^{|A|}$$

the sum being taken over all edge-subgraphs $< A >$ *of* G *whose rank is* i.

Corollary 3.7. *For Ising model,* $\alpha = \varepsilon$ *and* $\beta = \varepsilon^{-1}$, *thus:*

$$\widetilde{Z_G} = (\frac{1}{\varepsilon})^{|E|} \sum_{A \subset E} (\varepsilon^2 - 1)^{|A|} \left(\frac{1}{p}\right)^{r(X)}$$

Note: this is not a polynomial on ε and p except some special cases.

3.3 Expanding from Coloring model

In this case we have $\alpha' = 0$ and $\beta' = 1$ for model C. For general model $M = (\alpha, \beta)$, the shifting function is :

$$c = \beta - \alpha \qquad d = \alpha.$$

Therefore we have:

Lemma 3.8.

$$\widetilde{Z_G^M} = \alpha^{|E|} \sum_{A \subset E} \left(\frac{\beta}{\alpha} - 1\right)^{|A|} \widetilde{Z_X^C}$$

For local constant model, we have

Corollary 3.9.

$$2^{k(G) - |V(G)|} = 2^{-r(G)} = \sum_{A \subset E} (-1)^{|A|} \widetilde{Z_X^C}$$

Question: what's the combinatorical meaning of $\widetilde{Z_X^C}$?

For Euler model, it becomes:

Corollary 3.10.

$$\widetilde{Z_G^E} = \sum_{A \subset E} (-2)^{|A|} \widetilde{Z_X^C}$$

For Ising model, we get:

Corollary 3.11.

$$\widetilde{Z_G} = \varepsilon^{|E|} \sum_{A \subset E} \left(\varepsilon^{-2} - 1\right)^{|A|} \widetilde{Z_X^C}$$

3.4 The $\alpha - \beta$ plane

4 Other models

4.1 Indeterminant model

Tutte Polynomial One may also choose for Ω the polynomial ring $\mathbb{R}[v]$. Then define $w(s,t) = 1 + v$ if $s = t \in S$ and $w(s,t) = 1$ otherwise. The corresponding partition function can be written here as:

$$Z_G(q,v) = \sum_{\sigma} \prod_{(x,y)\in E(G)} (1 + v\delta_{\sigma(x),\sigma(y)}).$$

It depends polynomially on both q and v, and $Z_G(q,v)$ is the *dichromatic polynomial* of G. It is related to the usual *Tutte Polynomial* $T_G(q,v)$ by

$$Z_G(q,v) = q^{k(G)} v^{|V(G)|-k(G)} T_G(1 + qv^{-1}, 1 + v)$$

and

$$T_G(q,v) = (q-1)^{-k(G)} (v-1)^{-|V(G)|} Z_G((q-1)(v-1), v-1).$$

Another example Now consider $S = \{0,1\}$, $\Omega = \mathbb{Z}[T]$, $w(s,t) = f(s)f(t)$ where $f(0) = 1$ and $f(1) = T$. For any graph G, denote by $n_j(G)$ the number of vertices $x \in V(G)$ of degree j; in particular

$$\sum_{j\geq 0} n_j(G) = V(G).$$

Then

$$Z_G = 2^{n_0(G)}(1+T)^{n_1(G)}(1+T^2)^{n_2(G)}\cdots$$

and, from this partition function, one may read the sequence of the $n_j(X)$'s.

Separation of simple graphs by spin models We suppose now that S, which we prefer to write here as H, is itself a set of vertices of a simple graph H. Let Ω be $\mathbb{Z}$, and denote by $(w(s,t))_{s,t\in H}$ the adjacent matrix of the graphs H. (In fact, the S in the above example can be understood as a complete graph with loops on each vertex.) Given a finite simple graph G and a state $\sigma : G \to H$, the product $\prod_{xy\in E(G)} w(\sigma(x), \sigma(y))$ is 1 when σ is a homorphism of graphs and 0 otherwise. So Z_G is the cardinate of the set of homomorphisms from G to H. For example, when $H = K_t$, $Z_G = P_G(t)$.

Denote by $a_1, \cdots, a_r$ the edges of S, and let Ω be the polynomial ring

$$\mathbb{Z}[T_1, \cdots, T_r].$$

Define $w(s,t) = T_j$ if $(s,t) = t_j$ and $w(s,t) = 0$ if (s,t) is not an edge. The coefficient of the monomial $\prod_{1 \le j \le r} T_j$ in Z_G is the number of isomorphisms from G to H.

Proposition 4.1. *Let G_1 and G_2 be two non-isomorphic simple graphs. There exists a spin model M such that $Z_{G_1}^M \neq Z_{G_2}^M$.*

Proof. Obvious from the last example. □

4.2 Biggs Model

Biggs Model (When moreover $\Omega = \mathbb{C}$, the models which follow are called "interaction model" following Biggs).

Choose a ring Ω. Let S be a finite ring, consider a function $i : S \to \Omega$ which is even, namely which satisfies $i(-s) = i(s)$ for all $s \in S$, and set $w(s,t) = i(t-s)$. Then

$$Z_G = \sum_{\sigma} \prod_{xy \in E(G)} i(\sigma(y) - \sigma(x)).$$

This shows that Bigg's models reduce precisely to Potts' models when the function i is constant on $S - \{0\}$.

Example 7. *Circular coloring* A (p/q)*-coloring* of a graph G is a function $c : V(G) \to \{0, 1, \cdots, p-1\}$ such that $uv \in E(G)$ implies $q \le |c(u) - c(v)| \le p - q$. In other words, adjacent vertices receive colors that differ by at least q modulo p.

When $i(0) = 0$ and $i(s) = 1$ otherwise, Biggs's models reduce to proper coloring model, while circular coloring is a specification of $i(s) = 0$ when $|s| < q$ and $i(s) = 1$ otherwise. From this it's easy to know circular colorings are the generalization of proper colorings.

5 Appendix

5.1 The proof of Theorem 3.1

Proof. Fixed a state ϕ, we have:

$$\begin{aligned}
hom_\phi^M(G,H) &= \prod_{uv\in E(G)} w(\phi(u),\phi(v)) \\
&= \prod_{uv\in E(G)} t_{\alpha,\beta}(\phi(u),\phi(v)) \\
&= \prod_{uv\in E(G)} t_{c\alpha'+d,c\beta'+d}(\phi(u),\phi(v)) \\
&= \prod_{uv\in E(G)} (c\alpha'+d)\delta_{\phi(u),\phi(v)} + (c\beta'+d)(1-\delta_{\phi(u),\phi(v)}) \\
&= \prod_{uv\in E(G)} ct_{\alpha',\beta'}(\phi(u),\phi(v)) + d \\
&= \sum_{A\subset E} d^{|E-A|}c^{|A|} \prod_{uv\in A} t_{\alpha',\beta'}(\phi(u),\phi(v)) \\
&= d^{|E|}\sum_{A\subset E} (\frac{c}{d})^{|A|} \prod_{uv\in A} t_{\alpha',\beta'}(\phi(u),\phi(v))
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
Z_G^M &= \sum_\phi hom_\phi^M(G,H) \\
&= \sum_\phi d^{|E|}\sum_{A\subset E} (\frac{c}{d})^{|A|} \prod_{uv\in A} t_{\alpha',\beta'}(\phi(u),\phi(v)) \\
&= d^{|E|}\sum_\phi\sum_{A\subset E} (\frac{c}{d})^{|A|} \prod_{uv\in A} t_{\alpha',\beta'}(\phi(u),\phi(v)) \\
&= d^{|E|}\sum_{A\subset E} (\frac{c}{d})^{|A|} \sum_\phi\prod_{uv\in A} t_{\alpha',\beta'}(\phi(u),\phi(v)) \\
&= d^{|E|}\sum_{A\subset E} (\frac{c}{d})^{|A|} q^{|V(G)|-|V(X)|} Z_X^{M'} (\spadesuit)
\end{aligned}$$

where Step ($\spadesuit$) is from the fact that each state ϕ on G induces a state on $X =< A >= G|_A$, and each state on X is the restriction of $q^{|V(G)|-|V(X)|}$ states on G. □

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