Open Problems from the DocCourse

This list contains various open problems which were mentioned in the course. Some may be possible research topics; others I think are very difficult! A couple at the end were not mentioned in the course but are related to topics that were discussed.

1. Problems on derangements, especially of prime power order.

(a) Find, if possible, an elementary proof (that is, not using the Classification of Finite Simple Groups) of the *Fein–Kantor–Schacher theorem*:

> If n is a transitive permutation group of degree $n > 1$, then G contains a derangement (an element with no fixed points) with prime power order.

(b) *Isbell's Conjecture* from 1960 is the following:

> There is a function $f(p,b)$ such that, if $n = p^a \cdot b$ with p a prime not dividing b and $a > f(p,b)$, then any transitive permutation group of degree n contains a derangement of p-pwer order.

Prove this.

(c) Find "improvements" to the *Cameron–Cohen theorem*:

> If G is a transitive permutation group of degree $n > 1$, then the proportion of derangements in G is at least $1/n$.

Of course the bound is best possible; but if G is not sharply 2-transitive then improvements are possible. Some results on this are known.

2. Is it true that, with only a few exceptions, the cycle index of a transitive permutation group G (ignoring the factor $1/|G|$) is irreducible over $\mathbb{Z}$?

3. Let G be an oligomorphic permutation group on Ω, in which $f_n(G)$ is the number of orbits of G on n-element subsets of Ω.

(a) For which primitive groups G does $(f_n(G))$ grow no faster than exponentially? In such cases, do we always have

$$f_n(G) \sim An^b c^n$$

for constants a,b,c?

(b) Can one prove the existence of limits such as

$$\lim_{n\to\infty} \frac{\log f_n(G)}{\log n}, \qquad \lim_{n\to\infty} \frac{\log\log f_n(G)}{\log n}, \qquad \lim_{n\to\infty} \frac{\log f_n(G)}{n}?$$

(c) If G has no finite orbits, is it true that the algebra A^G is an integral domain and that the constant function $e \in V_1^G$ is prime in A^G?

(d) Is there an absolute constant C such that if G is primitive and $f_n(G) = f_{n+1}(G)$, then $n \le C$? (We have $n \le 6$ in all known examples.)

4. Let G be a subgroup of the automorphism group of a matroid M. Is there a polynomial which specialises to the number of G-orbits on all (or most) of the "standard" specialisations of the Tutte polynomial of G?

5. Can one develop some theory for the structure and geometry of bases of a more general class of groups than IBIS groups? [For example, groups where all minimal bases have the same cardinality, or those where the sizes of irredundant bases can differ by at most 1.]

6. Can one characterise homogeneous structures M over finite relational languages in terms of the growth rates of their orbit-counting sequences $(f_n(G))$ or $(F_n(G))$, where $G = \mathrm{Aut}(M)$? [Growth bounded by the exponential of a polynomial is necessary but not sufficient. Is growth strictly slower than the exponential of a quadratic polynomial sufficient for $(f_n(G))$?]

7. Let $G_{n,k}$ be the permutation group induced by S_n on the set of k-subsets of $\{1,\ldots,n\}$. Let $d(n,k)$ be the proportion of derangements in $G_{n,k}$. It is known that $d(n,k)$ tends to a limit $a(k)$ as $n \to \infty$ (for fixed k); for example, $a(1) = \mathrm{e}^{-1}$ and $a(2) = 2\mathrm{e}^{-3/2}$.

(a) Find an efficient way to calculate $a(k)$.

(b) Does $a(k)$ tend monotonically to 1 as $k \to \infty$?

8. Let s_n be the number of subgroups of the symmetric group S_n, and a_n the largest antichain in the subgroup lattice. What can be said about the asymptotics of s_n/a_n? [This number does not exceed the size of the longest chain of subgroups in S_n, which is known to be $\sim 3n/2$ (an exact formula is known).]