Problems from the DocCourse: Day 3

Problems

1. (a) Prove that, if there exists a finite transitive permutation group which contains no derangement of prime power order, then there exists a primitive simple permutation group with this property.

[Hint: If G has the property and has a nontrivial congruence, then the permutation group induced by G on the set of congruence classes has the property. Also, if G has the property, then so does any transitive subgroup.]

(b) Show that if G is a primitive permutation group containing no derangement of prime power order, then the stabiliser of a point is a maximal proper subgroup of G which intersects every conjugacy class of elements of prime power order in G.

(c) Prove that A_5 (the alternating group of degree 5) does not contain a maximal subgroup which meets every conjugacy class of elements of prime power order.

2. (a) Let G be a transitive permutation group of degree $n = p^a \cdot b$ where a does not divide b. Prove that the orbits of a Sylow p-subgroup of G have size at least p^a.

(b) Prove that, if P is a transitive p-group, then more than $(p-1)|P|/p$ elements of P are derangements. [Hint: Show that P has an intransitive subgroup of index p.] Deduce that a p-group with fewer than p orbits contains a derangement.

(c0 Hence show that if G is a transitive permutation group of degree $n = p^a \cdot b$ where $a > 0$ and $b < p$, then G contains a derangement of order a power of p.

From the book

1.31^{**}, 1.33^{*}, 4.3^{*}, 4.7^{*}, 4.22^{**}.