Problems from the DocCourse: Day 5

Two problems from Antonio Machì

1. A *descent* in a permutation g in the symmetric group S_n (on the set $\{1,2,\ldots,n\}$) is a point i such that $ig \leq i$; it is a *strict descent* if $ig < i$.

Prove that, if a subgroup G of S_n has h orbits, then the average number of descents of a permutation in G is $(n+h)/2$, and the average number of strict descents is $(n-h)/2$. Deduce the Orbit-Counting Lemma.

2. A combinatorial proof of Hurwitz's Theorem. You don't need to know anything about maps or Riemann surfaces!

(a) Let $z(g)$ be the number of cycles of a permutation $g \in S_n$, and $t(g)$ the minimum number of transpositions whose product is g. Prove that

$$z(g)+t(g) = n.$$

(b) Prove that, if $t_1,\ldots,t_k$ are transpositions which generate a transitive subgroup of S_n, then $k \geq n-1$. If, further, $t_1 \cdots t_k = 1$, then $k \geq 2n-2$ and k is even. [**Hint**: Think of the t_i as edges of a graph.]

(c) Hence show that, if $g_1,\ldots,g_m$ generate a transitive subgroup of S_n, then

$$z(g_1)+\cdots+z(g_m) \leq (m-1)n+1.$$

If, further, $g_1 \cdots g_m = 1$, then

$$z(g_1)+\cdots+z(g_m) \leq (m-2)n+2,$$

and the difference of these two quantities is even.

(d) How should the preceding result be modified if the group generated by $g_1,\ldots,g_m$ has a prescribed number p of orbits?

(e) Suppose that g_1,g_2,g_3 generate a regular subgroup G of S_n, and $g_1g_2g_3 = 1$. Let $z(g_1)+z(g_2)+z(g_3) = n+2-2g$. Prove *Hurwitz's Theorem*:

If $g \geq 1$, then the order of G is at most $84(g-1)$.

Construct an example meeting the bound when $g = 3$.

Hint: If $|G| = n$ and g_i has order n_i for $i = 1,2,3$, show that

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3} = 1-\frac{2(g-1)}{n}.$$

Problems on homogeneous structures

1. Let $\mathcal{G}_n$ be the class of finite graphs containing no complete subgraph on n vertices. Prove that $\mathcal{G}_n$ has the amalgamation property. Let H_n be the Fraïssé limit of this class, and $G_n = \mathrm{Aut}(H_n)$. (The graphs H_n were first constructed by Henson.)

Prove that, if $n = 3$, then the stabiliser of a vertex v in G_n acts highly transitively on the set of neighbours of v, but contains no finitary permutation.

Prove that, if $n > 3$, then the stabiliser of a vertex v in G_n, acting on the set of neighbours of v, is isomorphic to a subgroup of G_{n-1}.

2. Prove that the class of finite bipartite graphs does not have the amalgamation property.

Let $\mathcal{B}$ be the class of finite bipartite graphs with a distinguished bipartite block. Show that $\mathcal{B}$ has the amalgamation property. Let B be its Fraïssé limit, and $G = \mathrm{Aut}(B)$. Prove that G has two orbits on the set of vertices of B, and is highly transitive on each orbit but contains no finitary permutation.

3. This exercise is due to Sam Tarzi.

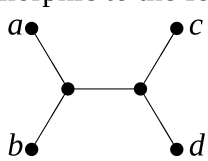
Let L be the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$. (If you know about lattices, do this question for an arbitrary lattice in $\mathbb{R}^d$.)

Given a finite set S of points of L, and a positive real number r, prove that there is a point $v \in L$ such that the Euclidean distances $||v-x||$, for $x \in S$, are all different and all greater than r.

Now let $(d_1, d_2, \ldots)$ be the list of all distances between pairs of points of L. Define a graph on the vertex set L by deciding independently at random, for each i, whether all pairs of points at distance d_i are edges or all are non-edges. Show that, with probability 1, this graph is isomorphic to the countable random graph R.

Deduce that the isometry group of L is a subgroup of $\mathrm{Aut}(R)$.

4**. A *boron tree* is a finite tree in which all vertices have degree 1 or 3. Let $\mathcal{X}$ be the class of finite relational structures with a quaternary relation (written $(ab|cd)$) defined as follows: the points of the structure are the leaves of a boron tree; the relation $(ab|cd)$ holds if and only if a, b, c, d are all distinct and the paths joining them form a tree homeomorphic to the following:



Prove that $\mathcal{X}$ has the amalgamation property. If X is its Fraïssé limit, and $G = \mathrm{Aut}(X)$, prove that G is 3-transitive but not 4-transitive, and is 5-set-transitive but not 6-set-transitive.