Problems from the DocCourse: Day 8

Orbits on k-sets

1. Let k be a positive integer and Ω a set with $|\Omega| \geq 2k+1$ (possibly infinite). Let $\binom{\Omega}{k}$ be the st of k-subsets of Ω, and V_k the vector space of functions from $\binom{\Omega}{k}$ to a fixed field F of characteristic zero. Define a map $T : V_k \to V_{k+1}$ by the rule

$$(Tf)(A) = \sum_{\alpha \in A} f(A \setminus \{\alpha\})$$

for $|A| = k+1$. Prove that T is one-to-one.

[Hint: Show that it is enough to prove this in the case where $|\Omega| = 2k+1$. Show that this case is equivalent to the assertion that, if $|\Omega| = 2k+1$ and $S : V_k \to V_k$ is defined by

$$(fS)(A) = \sum_{|B|=k, A \cap B = \emptyset} f(B),$$

then S is invertible. Prove this.]

Deduce that the numbers $f_k(G)$ of an oligomorphic group on k-sets satisfy $f_k(G) \leq f_{k+1}(G)$.

2. Suppose that Ω is infinite (or finite and sufficiently large). Suppose that the 2-subsets of Ω are coloursed with r colours (all of which are used) so that only r distinct colour schemes of $(k+1)$-sets occur. Prove that either

- there is a colour which occurs at most once at each vertex, or
- there is a colour for which the edges of that colour form a complete bipartite graph.

[Harder problem: How large does "sufficiently large" have to be?]

Deduce that, if G is an infinite primitive permutation group and $f_2(G) = f_3(G)$, then G is 3-set-transitive.

3. Let U be a subspace of the space of $m \times n$ complex matrices which contains no matrix of rank 1. Prove that $\dim(U) \leq (m-1)(n-1)$.

Hence show that if

$$A = \bigoplus_{i \geq 0} V_i$$

is a graded algebra over $\mathbb{C}$ (that is, $V_0 = \mathbb{C} \cdot 1$ and $V_i \cdot V_j \subseteq V_{i+j}$) which is an integral domain (i.e. has no divisors of zero), then

$$\dim(V_{i+j}) \geq \dim(V_i) + \dim(V_j) - 1.$$

Deduce that, if G is an oligomorphic permutation group for which the graded algebra A^G is an integral domain, then

$$f_{i+j}(G) \geq f_i(G) + f_j(G) - 1.$$

Find an example where equality holds for all i and j.