Association schemes and permutation groups

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Abstract

Every permutation group which is not 2-transitive acts on a nontrivial coherent configuration, but the question of which permutation groups G act on nontrivial association schemes (symmetric coherent configurations) is considerably more subtle. A closely related question is: when is there a unique minimal G-invariant association scheme? We examine these questions, and relate them to more familiar concepts of permutation group theory (such as generous transitivity) and association scheme theory (such as stratifiability).

Our main results are the determination of all regular groups having a unique minimal association scheme, and a classification of groups with no non-trivial association scheme. The latter must be primitive, and are either 2-homogeneous, almost simple, or of diagonal type. The diagonal groups have some very interesting features, and we examine them further. Among other things we show that a diagonal group with non-abelian base group cannot be stratifiable if it has ten or more factors, or generously transitive if it has nine or more; and we characterise the quaternion group Q_8 as the unique non-abelian group T such that a diagonal group with eight factors T is generously transitive.

1 Association schemes and coherent configurations

A *coherent algebra*, or *cellular algebra*, is an algebra of $n \times n$ complex matrices which has a basis $\{B_0, B_1, \ldots, B_t\}$ consisting of matrices with entries 0 and 1 satisfying the following conditions:

(a) $B_0 + B_1 + \cdots + B_t = J$, where J is the all-1 matrix;

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(b) there is a subset of $\{B_0,\dots,B_t\}$ with sum I, the identity matrix;
(c) the set $\{B_0,\dots,B_t\}$ is closed under transposition.

Since these matrices span an algebra, we have

(d) $B_iB_j = \sum_{k=0}^{t} b_{ij}^k B_k$,

where the b_{ij}^k are complex numbers. The algebra is called *homogeneous* if the subset referred to in (b) contains just one element, which we take to be $B_0 = I$.

Any $n\times n$ zero-one matrix can be regarded as the characteristic function of a subset of $\Omega\times\Omega$, where $\Omega = \{1,\dots,n\}$. Condition (a) says that the matrices $B_0,\dots,B_t$ correspond to a partition of Ω^2, and the other conditions can be translated into combinatorial statements about this partition. Thus, if the parts are $C_0,\dots,C_t$, then a subset of these sets partitions the diagonal; the transpose (or converse) of each part is another part; and, if $(x,y)\in C_k$, then the number of points z for which $(x,z)\in C_i$ and $(z,y)\in C_j$ is equal to b_{ij}^k, independent of the choice of (x,y). (So, incidentally, we see that the numbers b_{ij}^k are non-negative integers.) Such a combinatorial object is called a *coherent configuration*.

Conversely, any coherent configuration gives rise to a coherent algebra.

Given any permutation group G on Ω, we obtain a coherent configuration $C(G)$ whose classes are the orbits of G on Ω^2. Its coherent algebra, which we will denote by $K(G)$, is the *centraliser algebra* of G, the algebra of all matrices commuting with the permutation matrices arising from elements of G. The dimension of $K(G)$ is equal to the number of orbits of G on Ω^2: this number is called the *rank* of the permutation group G.

Note that $K(G)$ is homogeneous if and only if G is transitive, and $K(G)$ has the smallest possible dimension (namely 2) if and only if G is 2-transitive.

An *association scheme* is a coherent configuration in which each of the sets C_i is self-converse (closed under reversing the order of the elements in each pair). It is easy to show that an association scheme is homogeneous, so that $B_0 = I$. The classes $C_1,\dots,C_t$ can now be identified with sets of unordered pairs (2-element subsets of Ω), which partition the set of all 2-element subsets of Ω. The coherent algebra of an association scheme is called its *Bose–Mesner algebra*. Since all its matrices are symmetric, it is commutative.

A trivial example of a Bose–Mesner algebra is spanned by the two matrices I and $J-I$. Any other such algebra or association scheme is called *non-trivial*. Note that $C(G)$ is trivial if and only if G is 2-transitive.

We say that a permutation group G on Ω *preserves* an association scheme or coherent configuration if all the sets C_i in the partition are fixed setwise by G. This is equivalent to requiring that permutation matrices corresponding to the elements of G commute with all the basis matrices B_i. The *automorphism group* of the configuration is the group of permutations which preserve it. We will say that an association scheme or coherent configuration *admits* the permutation group G if G is contained in its automorphism group.

The configuration $C(G)$ is the finest one admitting G (in the sense of the partial order on partitions of Ω^2). However, it is less trivial to decide whether a group preserves a non-trivial association scheme, or indeed a unique minimal association scheme. These are the questions we attack in this paper.

We call a group *AS-free* if it preserves no non-trivial association scheme. One question we consider is:

Question 1 *Which groups are AS-free?*

If G is 2-transitive, then it has just two orbits on Ω^2, with characteristic functions I and $J-I$, and clearly it is AS-free. So our problem is more general than the classification of 2-transitive groups. In fact, we will describe the socle of an AS-free group, thus generalising Burnside's theorem (asserting that the socle of a 2-transitive group is either elementary abelian or simple).

Bailey [2] showed that, if association schemes are regarded as partitions of $\Omega \times \Omega$, and ordered by refinement (i.e. $F_1 \preceq F_2$ if F_1 refines F_2), then the supremum of two association schemes is always an association scheme, but the infimum need not be an association scheme. Moreover, if there is any association scheme below F_1 and F_2, then there is a unique "largest" such (which may be smaller than the infimum of F_1 and F_2 as partitions).

This suggests the question:

Question 2 *For which transitive permutation groups G is it true that there is a unique finest G-invariant association scheme?*

Call a group *AS-friendly* if it has this property. Note that an AS-free group is AS-friendly, since for an AS-free group G the trivial association scheme is the unique G-invariant scheme.

Let $\mathcal{A}$ and $\mathcal{B}$ be homogeneous coherent configurations on sets Γ and Δ, with adjacency matrices $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ respectively, where $A_0 = B_0 = I$. The operations of *crossing* and *nesting* are defined to give rise to homogeneous coherent configurations $\mathcal{A} \times \mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ respectively, with basis matrices

- $\{A_i \otimes B_j : i \in I, j \in J\}$ (for $\mathcal{A} \times \mathcal{B}$), and

- $\{A_i \otimes J : i \in I \setminus \{0\}\} \cup \{I \otimes B_j : j \in J\}$ (for $\mathcal{A}/\mathcal{B}$).

Their automorphism groups are respectively the direct product $\mathrm{Aut}(\mathcal{A}) \times \mathrm{Aut}(\mathcal{B})$ and the wreath product $\mathrm{Aut}(\mathcal{B})\,\mathrm{wr}\,\mathrm{Aut}(\mathcal{A})$ (in the actions of the direct and wreath product on the product of the two sets Γ and Δ). If $\mathcal{A}$ and $\mathcal{B}$ arise from groups, so do the crossed and nested configurations. Also, if $\mathcal{A}$ and $\mathcal{B}$ are association schemes, so are the crossed and nested configurations.

In this paper, the action of a direct or wreath product of permutation groups is always that on the product set, as above. The direct product is thus a subgroup of the wreath product. Note that the direct and wreath product of transitive groups are transitive.

2 AS-friendly permutation groups

There are a few easy things to say about AS-friendly groups. First, note that there is an obvious candidate for a minimal association scheme, namely the set of symmetrised orbitals: take the partition $C(G)$ of Ω^2, and merge each non-symmetric part with its converse. Of course, this is not in general an association scheme. Following Bailey [1], we call a permutation group *stratifiable* if the symmetrised orbitals form an association scheme. (The reason for the name will be given later.) Thus, a stratifiable group is AS-friendly.

Of course, if every orbital is symmetric, then $C(G)$ is itself an association scheme. A permutation group is called *generously transitive* if this condition holds (that is, if every pair of distinct points of Ω can be interchanged by some element of G).

A matrix representation of G over the real numbers is said to be *real-multiplicity-free* if all its real-irreducible constituents are pairwise inequivalent. Finally, an irreducible character of G is of *real*, *complex* or *quaternionic* type according as its Frobenius–Schur index is $+1$, 0 or -1.

Now Bailey [1] showed the following result. For completeness we give the important part of the proof.

Theorem 1 *For a finite transitive permutation group G, the following conditions are equivalent:*

(a) the symmetrised orbitals of G form an association scheme;
(b) the symmetric matrices in $K(G)$ form a subalgebra;
(c) the permutation representation of G is real-multiplicity-free;
(d) the complex irreducible constituents of the permutation character of G either have multiplicity 1, or have multiplicity 2 and quaternionic type.

PROOF. Conditions (a) and (b) are clearly equivalent; and (c) and (d) are equivalent by general representation theory. We show that (c) and (d) imply (a). So assume that (c) and (d) hold. The matrices of the symmetrised orbitals span the space of symmetric matrices in the centraliser algebra over the real numbers. We have to show that this space is a subalgebra. For this, it suffices to prove that it is commutative; for, if two symmetric matrices commute, then their product is symmetric.

If χ has real, complex or quaternionic type respectively, then the real centraliser algebra of χ, $\chi+\overline{\chi}$, or 2χ respectively is isomorphic to the real, complex, or quaternion division ring. Thus, there is a real orthogonal matrix P which transforms the centraliser algebra into a direct sum of copies of these division rings in their usual real representations, of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for the complex numbers and

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

for the quaternions. The symmetrised basis matrices remain symmetric on transformation by P, so by inspection they must be diagonal in each component. So they commute, and we are done. $\square$

Thus, in the terminology introduced above, G is stratifiable if and only if the permutation representation is real-multiplicity-free. The *strata* are the subspaces of $\mathbb{R}^\Omega$ affording real-irreducible representations of G, and play a role in analysis of variance.

The analogous result for generous transitivity is the following:

Proposition 2 *The permutation group G is generously transitive if and only if the permutation representation over $\mathbb{C}$ is multiplicity-free and all the irreducible constituents can be written over $\mathbb{R}$.*

We can now give some implications between the concepts we have defined and other concepts of permutation group theory. A permutation group G on Ω is

- 2-*homogeneous* if it is transitive on the 2-element subsets of Ω;
- *primitive* if it preserves no non-trivial equivalence relation on Ω.

Theorem 3 *The following implications hold between properties of a permutation group G:*

$$\begin{array}{ccccccc}
2\text{-}\textit{transitive} & \Rightarrow & 2\text{-}\textit{homogeneous} & \Rightarrow & \textit{AS-free} & \Rightarrow & \textit{primitive} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\textit{gen. trans.} & \Rightarrow & \textit{stratifiable} & \Rightarrow & \textit{AS-friendly} & \Rightarrow & \textit{transitive}
\end{array}$$

None of these implications reverses, and no further implications hold, save possibly that from "primitive" to "AS-friendly".

PROOF. The implications between the permutation group theoretic concepts are well known. We already noted that generous transitivity implies stratifiability.

If G is 2-homogeneous but not 2-transitive, it has three orbits on Ω^2, where the two non-trivial orbits form a converse pair; so G is AS-free and stratifiable.

We have seen that both "AS-free" and "stratifiable" imply "AS-friendly".

An example of a transitive group which is not AS-friendly will be given in the next section, where it will emerge naturally from the discussion. We do not know whether there is a primitive permutation group which is not AS-friendly, though it seems very likely that such groups exist.

Here is the proof that an AS-free group is primitive. Suppose that G is transitive but imprimitive. Let R be a non-trivial G-congruence. Let

$$\begin{aligned}
C_0 &= \{(\alpha,\alpha) : \alpha \in \Omega\}, \\
C_1 &= R \setminus C_0, \\
C_2 &= \Omega^2 \setminus R.
\end{aligned}$$

Then $\{C_0, C_1, C_2\}$ is an invariant association scheme admitting G as an automorphism group. (An association scheme of this type is called *group-divisible*, though this use of the word "group" is unconnected with the algebraic sense.)

Note, incidentally, that a group which is generously transitive and 2-homogeneous is 2-transitive, and a group which is stratifiable and AS-free is 2-homogeneous.

There are AS-free (and hence AS-friendly) groups which are not stratifiable. Indeed, there do exist almost simple primitive groups which are AS-free. This can be seen from the paper of Faradžev *et al.* [4]. These authors consider the following problem. Let G be an almost simple primitive permutation group, whose socle is a simple group of order at most 10^6 but not $\mathrm{PSL}(2,q)$. Describe the coherent subalgebras of $K(G)$. Table 3.5.1 on p.115 gives their results. In several cases, no non-trivial algebra consists entirely of symmetric matrices: such groups are of course AS-free. The smallest example is the group $\mathrm{PSL}(3,3)$, acting on the right cosets of $\mathrm{PO}(3,3)$ (a subgroup isomorphic to S_4), with degree 234. Other examples of AS-free groups in this list are M_{12}, degree 1320; J_1, degree 1463, 1540 or 1596; and J_2, degree 1800. (These examples were pointed out to us by Leonard Soicher.)

Finally, "AS-friendly" implies "transitive" by definition. □

We give a few more simple results about the class of AS-friendly groups. First there is a technical point to be addressed. An association scheme on Ω is a partition of Ω^2, but its automorphism group as we have defined it is not the stabiliser of the partition, but the group fixing all the parts of the partition. (Some authors use the term "weak automorphism" for a permutation fixing the partition, and "strong automorphism" for a permutation fixing all parts of the partition.) However, the following holds.

Lemma 4 *Let $\mathcal{C}$ be an association scheme on Ω, and G a permutation group on Ω which fixes $\mathcal{C}$ as a partition of Ω^2. Let $\tilde{\mathcal{C}}$ be the partition of Ω whose parts are the unions of the G-orbits on the parts of Ω. Then $\tilde{\mathcal{C}}$ is an association scheme and G a group of automorphisms of $\tilde{\mathcal{C}}$.*

PROOF. Note that the diagonal must be fixed by G, and that the parts of $\tilde{\mathcal{C}}$ are symmetric. Let I be an orbit of G on an index set for the parts of the partition $\mathcal{C}$, and let $C_I = \bigcup_{i\in I} C_i$ be the corresponding part of $\tilde{\mathcal{C}}$. For $(\alpha,\beta) \in C_k$, the number of points γ such that $(\alpha,\gamma) \in C_I$ and $(\gamma,\beta) \in C_J$ is

$$\sum_{i\in I, j\in J} b_{ij}^k,$$

and we have to show that this sum is independent of the choice of $k \in K$. But, given $k' \in K$, let $g \in G$ map k to k'; suppose that g maps α to α' and β to β', where $(\alpha',\beta') \in C_{k'}$. Since g permutes among themselves the indices in I and the indices in J, it is clear that the points counted in the sum for k are mapped to the points counted in the sum for k', and so their numbers are equal. □

Note that $\tilde{\mathcal{C}}$ is the finest association scheme which is coarser than $\mathcal{C}$ and admits G as a group of automorphisms.

Theorem 5 *(a) If a group has an AS-friendly subgroup, then it is AS-friendly.*
(b) The class of AS-friendly permutation groups is closed under wreath product.
(c) Let G be imprimitive; let Γ be a system of imprimitivity and Δ a block in Γ, and let H be the permutation group induced on Δ by its setwise stabiliser and K

the group induced on Γ by G, so that $G \le H \operatorname{wr} K$. If G is AS-friendly, then so are H and K.

(d) The same assertions hold with "stratifiable" or "generously transitive" in place of "AS-friendly".

PROOF. We will prove each stated result first and then argue for the modified versions required by (d).

(a) Let G_0 be a transitive subgroup of G, and suppose that G_0 is AS-friendly; let $\mathcal{A}_0$ be the minimal G_0-invariant association scheme. For $g \in G$, let $\mathcal{A}_0^g$ be the image of $\mathcal{A}_0$ under g. Then G permutes the collection $\{\mathcal{A}_0^g : g \in G\}$ of association schemes, and so preserves their supremum

$$\mathcal{A} = \bigvee_{g \in G} \mathcal{A}_0^g$$

as a partition. Let $\tilde{\mathcal{A}}$ be the association scheme which is obtained from $\mathcal{A}$ as in Lemma 4.

We claim that $\tilde{\mathcal{A}}$ is the minimal association scheme admitting G as a group of automorphisms. For let $\mathcal{B}$ be any association scheme admitting G. Then $\mathcal{B}$ admits G_0, so $\mathcal{A}_0 \preceq \mathcal{B}$. Applying any element $g \in G$, we find that $\mathcal{A}_0^g \preceq \mathcal{B}$. Hence $\mathcal{A} \preceq \mathcal{B}$. Since G is a group of automorphisms of $\mathcal{B}$, the remark after Lemma 4 shows that $\tilde{\mathcal{A}} \preceq \mathcal{B}$, as required.

Now let G_0 be a transitive subgroup of G, and suppose that G_0 is stratifiable. If G is not stratifiable, then some real irreducible constituent ϕ of the permutation representation has multiplicity greater than 1. But then, restricting to G_0, we find that a G_0-irreducible constituent of ϕ has multiplicity greater than 1, contrary to assumption.

The analogous result for generous transitivity is obvious.

(b) Let $G = H \operatorname{wr} K$, and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the minimal association schemes for H and K respectively. Let $\mathcal{A}$ be the nested scheme $\mathcal{A}_1/\mathcal{A}_0$. We claim that $\mathcal{A}$ is the minimal association scheme for G. (It is clearly G-invariant.)

Let $\mathcal{B}$ be any G-invariant association scheme. For any relation C of $\mathcal{B}$, we write $C = C^0 \cup C^1$, where C^0 consists of the pairs in C consisting of points in the same block, and C^1 of points in different blocks. Let

$$\mathcal{B}^* = \{C^\varepsilon : C \in \mathcal{C}, \varepsilon = 0, 1, C^\varepsilon \neq \emptyset\}.$$

We claim that $\mathcal{B}^*$ is an association scheme. Clearly it is G-invariant and imprimitive, with a block Δ, and is finer than $\mathcal{B}$. So the subscheme $\mathcal{B}_0$ on Δ and quotient scheme $\mathcal{B}_1$ on Γ satisfy

$$\mathcal{A}_0 \preceq \mathcal{B}_0, \qquad \mathcal{A}_1 \preceq \mathcal{B}_1, \qquad \mathcal{B}^* = \mathcal{B}_1/\mathcal{B}_0;$$

so $\mathcal{A} \preceq \mathcal{B}$, as required.

It remains to show that $\mathcal{B}^*$ is an association scheme.

Each basis matrix A_i of $\mathcal{B}$ has the form $A_i^0 \otimes I + J \otimes A_i^1$, where the two summands are the basis matrices of C^0 and C^1 into which the corresponding class C is split (or possibly zero). The matrices A_i^0 and A_i^1 have constant row and column sums k_i^0 and k_i^1 respectively. We have

$$(A_i^0 \otimes I + J \otimes A_i^1)(A_j^0 \otimes I + J \otimes A_j^1) = \sum b_{ij}^k (A_k^0 \otimes I + J \otimes A_k^1).$$

Hence for $i, j > 0$ (where $A_0 = I \otimes I$), by considering the diagonal blocks, we have

$$A_i^0 A_j^0 + ndJ = \sum b_{ij}^k A_k^0,$$

where d is a diagonal element of $A_i^1 A_j^1$, and $n = |\Gamma|$. We conclude that these diagonal elements are constant, and the matrices A_i^0 are the basis matrices of an association scheme $\mathcal{A}_0$. Then, considering the off-diagonal blocks, we have

$$k_j^0 A_i^1 + k_i^0 A_j^1 + n(A_i^1 A_j^1 - dI) = \sum b_{ij}^k A_k^1,$$

so the matrices A_i^1 are the basis matrices of a scheme $\mathcal{A}_1$. Clearly $\mathcal{B}^* = \mathcal{A}_1/\mathcal{A}_0$. This completes the proof.

If the schemes $\mathcal{A}_0$ and $\mathcal{A}_1$ are formed by the symmetrised orbitals of H and K, then the symmetrised orbitals of $H \operatorname{wr} K$ clearly form the scheme $\mathcal{A}_1/\mathcal{A}_0$. If all orbitals for H and K are symmetric, the same is true for $H \operatorname{wr} K$.

(c) Suppose that the hypotheses hold. Let $\mathcal{A}$ be the minimal association scheme for G. Since G preserves the two-class resolvable scheme $\mathcal{R}$ corresponding to the resolution Γ, we have $\mathcal{A} \preceq \mathcal{R}$. Hence $\mathcal{A}$ is an imprimitive association scheme, and induces a subscheme $\mathcal{A}_0$ on the block Δ.

We claim that $\mathcal{A}_0$ is the minimal association scheme for H. For let $\mathcal{B}$ be any association scheme for H. If $\mathcal{S}$ is the trivial scheme on Γ, then the nested scheme $\mathcal{S}/\mathcal{B}$ is preserved by $H \operatorname{wr} K$ and hence by G; so

$$\mathcal{A} \preceq \mathcal{S}/\mathcal{B},$$

so $\mathcal{A}_0 \preceq \mathcal{B}$.

Dually, let $\mathcal{A}_1$ be the quotient scheme of $\mathcal{A}$ on Γ, with classes

$$\overline{C} = \{(\Delta_1, \Delta_2) : C \cap (\Delta_1 \times \Delta_2) \neq \emptyset\}$$

for all classes C of $\mathcal{A}$. Then $\mathcal{A}_1$ is K-invariant, and we claim that it is the minimal K-invariant scheme. So let $\mathcal{C}$ be a K-invariant scheme. Then the nested scheme $\mathcal{C}/\mathcal{S}'$ is $H \operatorname{wr} K$-invariant, and hence G-invariant. So

$$\mathcal{A} \preceq \mathcal{C}/\mathcal{S}',$$

whence $\mathcal{A}_1 \preceq \mathcal{C}$, as required.

The proof for stratifiability is similar but easier, since the symmetrised orbitals of G consist either of pairs in the same block or of pairs in different blocks. The proof for generous transitivity is even easier. $\square$

Corollary 6 *If G has a regular abelian subgroup A, then G is stratifiable; and if G contains the holomorph of A, then G is generously transitive.*

PROOF. The first assertion follows from part (a) of the theorem (for stratifiable groups) and the fact that regular abelian groups are stratifiable (Bailey [1]: see the next section). For the second, A has an automorphism mapping any element to its inverse, so the hypothesis implies that G contains the group $\{x \mapsto ax^{\pm 1} : a \in A\}$ of permutations of A, and this group is already generously transitive. (The permutation $x \mapsto bcx^{-1}$ interchanges b and c.) $\square$

Similar positive results do not exist for direct product. We defer this until the next section, when counterexamples will arise naturally.

3 Regular permutation groups

The main result of this section is the characterisation of groups whose regular representation is AS-friendly. In particular, we will show that the properties "AS-friendly" and "stratifiable" coincide for regular groups.

The *inverse partition* of a group G is the partition whose parts are the sets $\{g, g^{-1}\}$ for $g \in G$. Also, a partition $F = \{F_0, \ldots, F_s\}$ of a group G is called a *blueprint* if the partition $\tilde{F} = \{\tilde{F}_0, \ldots, \tilde{F}_s\}$ of $G \times G$ given by

$$\tilde{F}_i = \{(x, y) : xy^{-1} \in F_i\}$$

is an association scheme on G. (Note that this association scheme is invariant under right translation by G.)

Theorem 7 *For a finite group G, the following five conditions are equivalent:*

(a) the regular action of G is AS-friendly;
(b) the inverse partition of G is a blueprint;
(c) the regular action of G is stratifiable;
(d) the complex irreducible characters of G either have degree 1, or have degree 2 and quaternionic type;
(e) either G is abelian, or $G \cong Q \times A$ where Q is the quaternion group of order 8 and A is an elementary abelian 2-group.

PROOF. (e) implies (d): If G is abelian then all its irreducible characters have degree 1. It is well-known that the quaternion group Q has four real characters of degree 1 and one quaternionic character of degree 2. Now all characters of an elementary abelian 2-group are real of degree 1; and to calculate character degrees and Frobenius–Schur indices for a direct product of groups, we multiply the corresponding numbers for the factor groups.

(d) implies (c): In the regular representation, the multiplicity of any character is equal to its degree. Now if χ is a complex irreducible character of G, then χ, $\chi+\overline{\chi}$, or 2χ is the character of a real irreducible representation, depending on whether χ is real, complex, or quaternionic. So G is stratifiable if and only if the real and complex irreducible constituents have multiplicity 1 and the quaternionic constituents have multiplicity 2 in the permutation representation. Now (d) is just the assumption that this holds for the regular representation.

(c) implies (a): By Theorem 1, if (c) holds, then the symmetrised orbitals form an association scheme, whose blueprint is clearly the inverse partition; this blueprint is also clearly minimal, so (c) also implies (b).

(a) implies (b): Suppose that the regular group G is AS-friendly. For any subgroup H of G, the cosets of H form a system of imprimitivity, such that the group induced on the block H is just H acting regularly. Thus G is a subgroup of $H \operatorname{wr} S_m$, where m is the index of H in G.

Now take H to be the cyclic group generated by an arbitrary non-identity element x of G. Then H is abelian, so (since we have already shown that (e) implies (b)) its inverse partition is a blueprint, giving rise to a cyclic association scheme $\mathcal{C}$. Thus, $H \operatorname{wr} S_m$ preserves the nested scheme $\mathcal{S}/\mathcal{C}$. Since G is a subgroup of $H \operatorname{wr} S_m$, it also preserves this scheme. Thus, if $\mathcal{M}$ is the minimal G-invariant scheme, then $\mathcal{M} \le \mathcal{S}/\mathcal{C}$. This means that $\{x, x^{-1}\}$ is a class in the blueprint M defining $\mathcal{M}$. Since x is arbitrary, M is the inverse partition.

(b) implies (e): We suppose that G is a group in which the inverse partition is a blueprint. We proceed in a number of steps. Note that, since the Bose–Mesner algebra of an association scheme is commutative, the elements $a+a^{-1}$ of the group algebra of G commute.

Step 1 For any $a,b \in G$, one of the following cases holds: (a) a and b commute; (b) a inverts b; (c) b inverts a; (d) $(ab)^2 = 1$.

For $ba \in \{ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1}\}$.

Step 2 Involutions in G commute.

For if u and v are involutions, these four cases coincide.

Step 3 Involutions in G are central.

For let u be an involution and a an arbitrary element with order greater than 2. Then either $u(a+a^{-1}) = v+w$, where v and w are involutions, or $u(a+a^{-1}) = b+b^{-1}$ for some b. In the first case, $ua = v$, whence $a = uv$ is an involution by Step 2, contrary to the case assumption. In the second case, $ua = b$ and

$$ua^{-1} = b^{-1} = a^{-1}u,$$

so $ua = au$.

Step 4 We can delete (d) in Step 1.

For in that case ab commutes with b by Step 3, so a and b commute.

Step 5 For any two elements a and b, either a and b commute, or each inverts the other.

For suppose that a and b do not commute, and (say) that a inverts b, that is, $ba = ab^{-1}$. Then ab and b do not commute, so either $a^{-1}aba = (ab)^{-1}$, or $(ab)^{-1}aab = a^{-1}$. In the first case, $ab^{-1} = ba = b^{-1}a^{-1}$, whence $bab^{-1} = a^{-1}$. In the second, $b^{-1}ab = a^{-1}$ follows immediately. So b also inverts a in either case.

Step 6 G is Hamiltonian, that is, every subgroup is normal.

This follows from the fact that all cyclic subgroups are normal (Step 5).

Step 7 Completion of the proof. By the theorem of Dedekind (see [6], Satz 7.12 on p. 308), either G is abelian, or $G = Q \times A \times B$, where Q is the quaternion group of order 8, A is an elementary abelian 2-group, and B is abelian of odd order. We have to show that $B = 1$. Let i, j, k be elements of order 4 in Q with $i^2 = j^2 = k^2 = ijk = z$, and take any $a, b \in B$. Then

$$(ia + zia^{-1})(jb + zjb^{-1}) = kab + ka^{-1}b^{-1} + zkab^{-1} + zka^{-1}b.$$

Since $k^{-1} = zk$, we see that $(ab)^{-1}$ is either ab^{-1} or $a^{-1}b$. This means that either $a^2 = 1$ or $b^2 = 1$, whence (since B has odd order) either $a = 1$ or $b = 1$. Since a and b are arbitrary, $B = 1$. □

An example of a transitive group which is not AS-friendly is given by the smallest regular group not covered by this theorem, namely the dihedral group of order 6. Let G be this group:

$$G = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1 \rangle.$$

Then

$$\{\{1\}, \{a, a^2\}, \{b\}, \{ab, a^2b\}\}$$

is a blueprint in G. The resulting association scheme is the 3×2 rectangle: the classes are "same row", "same column", and "neither". Similarly, if the three involutions are partitioned into sets of sizes 1 and 2 in any manner, the result is again an association scheme. But we cannot split the involutions into three singleton classes and get an association scheme, since they do not commute.

It is elementary that the direct product of generously transitive groups is generously transitive. However:

Corollary 8 *The classes of stratifiable groups and of AS-friendly groups are not closed under direct product.*

PROOF. The direct product of regular groups is regular. But the main theorem of this section shows that Q_8 acting regularly is stratifiable but $Q_8 \times Q_8$ acting regularly is not AS-friendly. □

The direct product of two groups of degree greater than 1 is obviously imprimitive. So none of the properties in the top row of Theorem 3 is closed under direct product.

4 AS-free permutation groups

We have seen that 2-homogeneous groups are AS-free. Are there any other transitive AS-free groups?

A permutation group is called *non-basic* if there is a bijection between Ω and Γ^{Δ} (the set of functions from Δ to Γ) for some finite sets Γ and Δ, which induces an isomorphism from G to a subgroup of $\mathrm{Sym}(\Gamma)\,\mathrm{wr}\,\mathrm{Sym}(\Delta)$ with the product action. This concept arises in the O'Nan–Scott classification of primitive permutation groups, see [3], p. 106.

Theorem 9 *Let G be a transitive AS-free group. Then G is primitive and basic, and is 2-homogeneous, diagonal or almost simple.*

PROOF. We have seen that an AS-free group is primitive (Theorem 3).

Now suppose that G is not basic. Then we can identify Ω with the set of m-tuples of elements taken from an alphabet Γ, where we identify Δ with the set $\{1,\ldots,m\}$. For $i=0,\ldots,m$, let C_i be the set of pairs of elements of Ω whose Hamming distance is i. This is an association scheme, the so-called *Hamming scheme*, and is preserved by G.

So G is basic. It follows from the O'Nan–Scott theorem that G is affine, diagonal, or almost simple. The affine groups contain abelian regular subgroups, and so are stratifiable; hence an affine group is AS-free if and only if it is 2-homogeneous. $\square$

5 Diagonal groups

In this section and the next two, we consider diagonal groups, not necessarily primitive. We define them in general, investigate when they are primitive and when they are generously transitive or stratifiable, and show that a primitive diagonal group whose socle has two or three simple factors is not AS-free.

The *diagonal group $D(T,n)$*, where T is a group and n a positive integer, is defined as the permutation group on the set

$$\Omega = T^n = \{[x_1,\ldots,x_n] : x_1,\ldots,x_n \in T\}$$

generated by the following permutations:

(a) the group T^n acting by right translation, that is, the permutations

$$[x_1,\ldots,x_n] \mapsto [x_1t_1,\ldots,x_nt_n]$$

for $t_1,\ldots,t_n \in T$;

(b) the automorphism group of T, acting coordinatewise, that is,

$$[x_1,\ldots,x_n] \mapsto [x_1^\alpha,\ldots,x_n^\alpha]$$

for $\alpha \in \mathrm{Aut}(T)$;

(c) the symmetric group S_n, acting by permuting the coordinates, that is,

$$\pi : [x_1,x_2,\ldots,x_n] \mapsto [x_{1\pi},x_{2\pi},\ldots,x_{n\pi}]$$

for $\pi \in S_n$;

(d) the permutation

$$\tau : [x_1,x_2,\ldots,x_n] \mapsto [x_1^{-1},x_1^{-1}x_2,\ldots,x_1^{-1}x_n].$$

We give another description of this group now and explain why elements of Ω are written in square brackets.

The diagonal group is usually defined, in the case where T is simple, as the group

$$G = T^{n+1}(\mathrm{Out}(T) \times S_{n+1})$$

acting on the set of right cosets of the subgroup

$$H = \mathrm{Aut}(T) \times S_{n+1}$$

by right multiplication, where the inner automorphisms are identified with the diagonal subgroup

$$\{(t,t,\ldots,t) : t \in T\}$$

of T^{n+1}. This is how diagonal groups arise in the O'Nan–Scott Theorem.

Each right coset of H in G has a unique coset representative of the form $(1,x_1,\ldots,x_n) \in T^{n+1}$; we denote this coset by $[x_1,\ldots,x_n]$. Now we check the actions of the elements of G. For $t_1,\ldots,t_n \in T$, we have

$$\begin{aligned}[x_1,\ldots,x_n](1,t_1,\ldots,t_n) &= H(1,x_1,\ldots,x_n)(1,t_1,\ldots,t_n)\\ &= H(1,x_1t_1,\ldots,x_nt_n)\\ &= [x_1t_1,\ldots,x_nt_n];\end{aligned}$$

so we obtain the permutations in (a). Any element of the base group is uniquely the product of an element with first coordinate 1 and a diagonal element; and we have

$$\begin{aligned}[x_1,\ldots,x_n](t,t,\ldots,t) &= H(1,x_1,\ldots,x_n)(t,t,\ldots,t)\\ &= H(t,x_1t,\ldots,x_nt)\\ &= H(1,t^{-1}x_1t,\ldots,t^{-1}x_nt)\\ &= [t^{-1}x_1t,\ldots,t^{-1}x_nt];\end{aligned}$$

so diagonal elements act by conjugation, as part of type (b). Clearly the outer automorphisms give the remainder of type (b). Permutations fixing the first coordinate clearly give us type (c). To generate S_{n+1}, we have to add in the transposition τ of the first two coordinates. We have

$$\begin{aligned}[x_1,x_2,\ldots,x_n]\tau &= H(1,x_1,x_2,\ldots,x_n)\tau\\ &= H(x_1,1,x_2,\ldots,x_n))\\ &= H(1,x_1^{-1},x_1^{-1}x_2,\ldots,x_1^{-1}x_n)\\ &= [x_1^{-1},x_1^{-1}x_2,\ldots,x_1^{-1}x_n],\end{aligned}$$

as in (d).

Our definition is computationally simpler and applies to any group T, not necessarily simple. Note that the number n in the notation $D(T,n)$ is one fewer than the number of direct factors isomorphic to T.

Note that the group $T^{n+1}\,\mathrm{Out}(T)$ generated by types (a) and (b) is normalised by S_{n+1} generated by types (c) and (d). Moreover, $S_{n+1} = S_n \cup S_n\tau S_n$, where τ is the transformation (d). So every combination of these transformations is either of the form abc or of the form $abc\tau c'$, where a,b,c,c' are of types (a), (b), (c) and (c) respectively. This further simplifies the job of computing the orbits of the diagonal group.

One of our goals is to decide which diagonal groups, if any, preserve no non-trivial association schemes (where the trivial scheme has just one associate class). First we establish conditions for a diagonal group to be primitive.

Proposition 10 *Let T be a non-abelian group and n a positive integer. The group $D(T,n)$ is primitive if and only if T is characteristically simple (that is, the direct product of isomorphic simple groups).*

PROOF. The subgroup T^n of $D(T,n)$ acts regularly, so the blocks of imprimitivity containing the identity are just the subgroups of T^n. Now $D(T,n)$ is primitive if and only if the only subgroups of T^n invariant under all permutations of types (b), (c) and (d) are 1 and T^n.

If U is a characteristic subgroup of T (one fixed by all automorphisms), then U^n is fixed by all permutations of types (b)–(d).

Conversely, suppose that T is characteristically simple, and that X is a non-trivial subgroup of T^n fixed by all permutations of types (b)–(d). Using types (b) and (c) we see that the projection of X onto each coordinate is the same characteristic subgroup of T, necessarily all of T. Now we are finished if $n = 1$, so assume that $n > 1$.

Now we observe that either $X = \{[t, \ldots, t] : t \in T\}$ or X contains an element of the form $[t, t^{-1}, 1, \ldots, 1]$ for $t \neq 1$. For if the first alternative does not hold, then T contains an element $[x, y, \ldots]$, and so also $[y, x, \ldots]$, with $x \neq y$; then we obtain $[xy^{-1}, yx^{-1}, 1, \ldots, 1]$.

The first case is immediately excluded by using the permutation (d).

Since T is non-abelian and characteristically simple, for any $t \in T$ with $t \neq 1$, every element of T is a product of images of t under $\mathrm{Aut}(T)$. So for any $u \in T$, there exists $u' \in T$ such that $[u, u', 1, \ldots] \in X$. If there is more than one such u' for some u, we find an element of X supported in just one coordinate, so that $X = T^n$, contrary to assumption. So $u' = u^\theta$ for some function $\theta : T \to T$, and it is clear that θ is an automorphism. Swapping coordinates shows that $\theta^2 = 1$. If $n > 2$, then also $[1, u^\theta, u, \ldots] \in X$, and so $[u, 1, u^{-1}, \ldots] \in X$; thus θ is inversion, which is impossible. So $n = 2$. Using the permutation (d) we find $[u^{-1}, u^{-1}u^\theta] \in T$, so that $(u^{-1})^\theta = u^{-1}u^\theta$. This gives $u^\theta = u^2$. But squaring is not an automorphism in a non-abelian group. $\square$

Now a characteristically simple group T is a direct product of isomorphic simple groups, say $T = T_1 \times \ldots \times T_m$, where $T_1, \ldots, T_m$ are isomorphic simple groups. Suppose that T is non-abelian. Then $\mathrm{Aut}(T) \cong \mathrm{Aut}(T_1) \,\mathrm{wr}\, S_m$. Now, in $D(T, n)$, the automorphisms of T act in the same way on each factor. In particular, S_m acts in the same way on the m copies of T_1 in each factor. From this it follows that

$$D(T, n) \cong D(T_1, n) \,\mathrm{wr}\, S_m,$$

so that $D(T, n)$ is non-basic. Moreover, if T is abelian, the diagonal group $D(T, n)$, which may or may not be primitive, is of affine type. For this reason, only the case where T is non-abelian simple arises in the O'Nan–Scott theorem.

For groups which are not characteristically simple, we have the following reduction.

Proposition 11 *Let $\mathcal{P}$ denote one of the properties "AS-friendly", "stratifiable", or "generously transitive". If $D(T, n)$ has property $\mathcal{P}$ and S is a characteristic subgroup of T, then $D(S, n)$ and $D(T/S, n)$ have property $\mathcal{P}$.*

PROOF. As we saw, $D(T,n)$ has a block of imprimitivity consisting of the n-tuples whose coordinates all lie in S. By Theorem 5(c), the group H induced on the block by its stabiliser and the group K induced on the set of translates of the block both have property $\mathcal{P}$. These groups are subgroups of $D(S,n)$ and $D(T/S,n)$ respectively: they contain the permutations of types (a), (c) and (d) but possibly not all the automorphisms, since not all automorphisms of S or T/S are necessarily induced by automorphisms of T. But, in any case, the result now follows from Theorem 5(a). $\square$

The converse of this result is false. If we take $T = S_3$ and $S = A_3$, then S and T/S are abelian, so the diagonal groups are generously transitive for all n; but T is non-abelian, so $D(T,n)$ fails to be generously transitive for some n, as we will see in the next section.

6 Generous transitivity and stratifiability of diagonal groups

We now examine when a diagonal group is generously transitive or stratifiable.

Consider the following three properties which a group T may have, where n is a positive integer.

(P1) Given any n-tuple $[t_1,\ldots,t_n]$ of elements of T, there is an automorphism of T which maps t_i to t_i^{-1} for $i = 1,\ldots,n$.
(P2) Given any n-tuple $[t_1,\ldots,t_n]$ of elements of T, there is an automorphism of T and a permutation of the entries of the n-tuple whose composition maps t_i to t_i^{-1} for $i = 1,\ldots,n$.
(P3) Given any n-tuple $[t_1,\ldots,t_n]$ of elements of T, either the conclusion of (P2) holds, or there is a composition of an automorphism of T, a permutation of the entries of the n-tuple, the transformation

$$\tau : [t_1,\ldots,t_n] \mapsto [t_1^{-1}, t_1^{-1}t_2,\ldots,t_1^{-1}t_n],$$

and another permutation of the entries, which maps t_i to t_i^{-1} for $i = 1,\ldots,n$.

Proposition 12 *(a) The conditions (P1), (P2), (P3) become successively weaker for given T and n (that is, each implies the next).*
(b) The conditions (P1) and (P2) become stronger as n increases, for given T. It is not clear if the same is true for (P3).
(c) Condition (P1) holds for all n if T is abelian.
(d) The diagonal group $D(T,n)$ is generously transitive if and only if (P3) holds for T and n.

PROOF. (a) and (b) are trivial, while (c) and (d) follow from our earlier remarks. Note that (P1) and (P2) correspond to generous transitivity of the subgroups $T^{n+1}.\mathrm{Out}(T)$ and $T^{n+1}.(\mathrm{Out}(T)\times S_n)$ of $D(T,n)$. $\square$

Here is a partial converse for (c).

Theorem 13 *For $i=1,2,3$, a group having property (Pi) for $n\geq 3i$ is abelian.*

PROOF. Suppose that T is non-abelian, and take $g,h\in G$ with $gh\neq hg$. Then there is no automorphism θ such that $g\theta=g^{-1}$, $h\theta=h^{-1}$, and $(gh)\theta=(gh)^{-1}$. For if $g\theta=g^{-1}$ and $h\theta=h^{-1}$, then $(gh)\theta=g^{-1}h^{-1}=(hg)^{-1}$.

(i) For $n\geq 3$, the n-tuple $[g,h,gh,1,\ldots 1]$ cannot be inverted by an automorphism.
(ii) For $n\geq 6$, take the n-tuple with one entry g, two entries h and $n-3$ entries gh. The different numbers of occurrences of the entries ensure that, if a combination of a permutation and an automorphism inverts this tuple, the automorphism must invert g, h and gh.
(iii) For $n\geq 9$, take the n-tuple with two entries g, three entries h and $n-5$ entries gh. Applying τ preceded and followed by automorphisms and permutations gives a tuple containing the identity, since each entry of the original tuple occurs more than once. If τ is not used, the argument is as in case (ii). $\square$

The value $n=3$ is best possible for (P1), as many examples show. The value $n=6$ is best possible for (P2): an example is the quaternion group Q_8, as we now show.

Write the elements of Q_8 as $z^\varepsilon, z^\varepsilon i, z^\varepsilon j, z^\varepsilon k$, with $i^2=j^2=k^2=ijk=z$, $z^2=1$, and $\varepsilon=0$ or 1. Note that, for any two generating pairs of elements, there is a (unique) automorphism mapping the first pair to the second.

Consider any n-tuple $[x_1,\ldots,x_n]$ which cannot be inverted.

(a) We can ignore occurrences of z^ε, since these elements are equal to their inverses and are fixed by all automorphisms.
(b) The numbers of occurrences of i and zi are unequal. For if they are equal, apply the automorphism $j\mapsto zj$, $k\mapsto zk$, $i\mapsto i$, and then swap the is and zis. Similarly for each of $z^\varepsilon j$ and $z^\varepsilon k$. In particular, at least one of each pair $\{i,zi\}$, $\{j,zj\}$ and $\{k,zk\}$ must occur.
(c) The numbers of occurrences of i and zi cannot be equal to the numbers of occurrences of j and zj. For suppose, say, that there are equally many is and js, and equally many zis and zjs. Apply the automorphism $i\mapsto zj$, $j\mapsto zi$, $k\mapsto zk$, and then swap the $z^\varepsilon i$s with the $z^\varepsilon j$s. In particular, at least two elements from two of the three inverse pairs must occur.

Now the only pattern with $n < 6$ would have, say, one of $\{i, zi\}$, two of $\{j, zj\}$ and two of $\{k, zk\}$. If we have j, zj or k, zk, point (b) is contradicted; otherwise point (c) is contradicted.

Note that the 6-tuple $[i, j, j, k, k, k]$ is of the type produced in the theorem.

In fact, for condition (P3), the number 9 can be reduced to 8.

Suppose that T is non-abelian; take $g, h \in T$ with $gh \neq hg$. We may also suppose that g does not have order 2. (If it does, and if h or gh has order greater than 2, use (h, g) or (gh, g) instead; if all three had order 2 then g and h would commute).

Consider the 8-tuple

$$[g, g, g^{-1}, h, h, h, gh, gh].$$

Clearly if a combination of permutation and automorphism inverts this tuple, then the automorphism must invert g, h and gh, which is impossible. So we must use τ. To avoid introducing the identity into the tuple, we must move g^{-1} to the first position before applying τ, giving

$$[g, g^2, g^2, gh, gh, gh, g^2h, g^2h].$$

Now we must have either $g^{-1} = g^2$, or $g^{-1} = g^2h$. The latter is impossible as it implies that $h = g^{-3}$, so g and h would commute. So $g^3 = 1$. Also, the automorphism must now map $g \mapsto g$, $gh \mapsto h^{-1}$, and $g^{-1}h \mapsto h^{-1}g^{-1}$, from which we deduce that $hgh^{-1} = g^{-1}$. So h has even order.

If h had order greater than 2, we could use h in place of g in the original argument to derive a contradiction. So $h^2 = 1$, and g and h generate S_3. But now it is easy to construct an 8-tuple which cannot be mapped to its inverse, for example,

$$[g, g, h, gh, gh, g^2h, g^2h, g^2h].$$

Corollary 14 *If the diagonal group $D(T, n)$ is generously transitive for $n \geq 8$, then T is abelian.*

Again the quaternion group Q_8 shows that this result is best possible. For consider any n-tuple, where $n = 6$ or $n = 7$. If one of the conditions we used earlier for (P2) holds (in particular, if $\{i, zi\}$, $\{j, zj\}$ and $\{k, zk\}$ all occur at most twice, or if they occur at most five times altogether), then we are done. If $\{i, zi\}$ occur at least three times, we may assume that one occurrence is in the first position; then applying τ gives a tuple containing $1, z$ at least twice, and the preceding case applies.

For $n = 7$, the following result shows that the quaternion group is the only possible example.

Theorem 15 *Let T be a non-abelian group such that $D(T,7)$ is generously transitive. Then $T \cong Q_8$.*

PROOF. Let $g,h,\in T$ with $gh \neq hg$. The group generated by g and h is non-abelian and thus at least one of g,h and gh must be of order greater than 2; without loss of generality $(gh)^2 \neq 1$. Consider the 7-tuple

$$[g,g,h,h,h,gh,(gh)^{-1}]$$

There are two cases for the transformation inverting this tuple:

(a) a transformation of type (P2); or
(b) $(\sigma_1\tau\sigma_2)\cdot\theta$ for some $\sigma_1,\sigma_2 \in S_7$ and $\theta \in \mathrm{Aut}(T)$, where τ is as defined before.

In (a), the automorphism θ is the map $g \mapsto g^{-1}$, $h \mapsto h^{-1}$, $gh \mapsto g^{-1}h^{-1}$. Thus, $g^{-1}h^{-1} = gh$ and therefore, $g^2h^2 = 1$. In (b), it is enough to look at the two possible σ_1's, namely $(1,6,5,4,3,2)$ and $(1,7,6,5,4,3,2)$. In the first case, the automorphism θ must be the map $gh \mapsto gh$ and $(gh)^{-2} \mapsto gh$. In the second case, the automorphism θ must be the map $gh \mapsto gh$ and $(gh)^2 \mapsto (gh)^{-1}$ or $gh \mapsto (gh)^{-1}$ and $(gh)^2 \mapsto gh$. Both cases lead to $(gh)^3 = 1$. Hence, if $gh \neq hg$ and (P3) is satisfied then $(gh)^2 \neq 1$ implies $(g^2h^2 = 1 \vee (gh)^3 = 1)$. By replacing (g,h) by pairs of noncommuting elements (g^{-1},gh) and then (gh,h^{-1}), we deduce 27 cases from the scheme below:

$$\begin{aligned}
&(gh)^2 = 1 \vee \quad g^2h^2 = 1 \quad \vee (gh)^3 = 1;\\
&\;h^2 = 1 \quad \vee g^{-2}(gh)^2 = 1 \vee \quad h^3 = 1;\\
&\;g^2 = 1 \quad \vee (gh)^2h^{-2} = 1 \vee \quad g^3 = 1.
\end{aligned}$$

To solve for the group generated by g and h with the given relations, a computer program is implemented using GAP [5]. This narrows down the choices to three possible groups, namely S_3, A_4 and Q_8. Now further choices of 7-tuples show that neither S_3 nor A_4 can satisfy (P3).

So we have shown that, for any g and h in T with $gh \neq hg$,

$$\langle g,h\rangle \cong Q_8.$$

Now, as in Steps 5 and 6 of Theorem 7, every subgroup of T is normal. By Dedekind's Theorem, T is isomorphic to $Q_8 \times A \times B$, where A is an elementary abelian 2-group and B is abelian of odd order. The following argument shows that $B = 1$. If $t \in B$

and $t \neq 1$ then, $(xt)(yt) \neq (yt)(xt)$. Thus $\langle xt, yt \rangle$ should be isomorphic to Q_8. But $(xt)^4 \neq 1$. Hence, $t = 1$ and B is trivial.

Any group isomorphic to $Q_8 \times A$ with $A \neq 1$ contains a subgroup

$$Q_8 \times C_2 = \langle x, y, z | x^4 = z^2 = 1, x^2 = y^2, y^{-1}xy = x^{-1}, xz = zx, yz = zy \rangle.$$

Consider the tuple $[z, xz, yz, yz, xyz, xyz, xyz]$. If a transformation in (P2) is used to invert it then the automorphism θ of $Q_8 \times C_2$ should be the map $z \mapsto z$, $xz \mapsto (xz)^{-1}$, $yz \mapsto (yz)^{-1}$. The entry xyz cannot be inverted as it is mapped to $(yxz)^{-1}$. On the other hand, if $(\sigma_1 \tau \sigma_2) \cdot \theta$ is used, then the following describes θ. If $\sigma_1 = (1)$ then θ should be the map $z \mapsto z$, $x \mapsto (xz)^{-1}$, $y \mapsto (yz)^{-1}$ and $xy \mapsto (yx)^{-1}$. This leads to a contradiction as xyz cannot be inverted. If $\sigma_1 = (12)$, then when τ is applied, there is no element of order 2 in the resulting tuple. Therefore, T cannot be isomorphic to $Q_8 \times A$ where A is non-trivial. $\square$

We now turn to the question of stratifiability. We begin with an observation. Let G be a group with a regular subgroup H. Then we can identify the set on which G acts with H, so that H acts by right multiplication. If A is an orbit of the stabiliser of the identity, then the paired orbit is A^{-1}. So G is stratifiable if and only if, for any orbits A and B of G_1, we have

$$(A + A^{-1})(B + B^{-1}) = (B + B^{-1})(A + A^{-1})$$

in the group ring, where we identify a set with the sum of its elements. In particular, a necessary condition is that, for any $a \in A$ and $b \in B$, we have $ba \in (A \cup A^{-1})(B \cup B^{-1})$.

Now define conditions (Q1), (Q2), (Q3) for any group T and positive integer n as follows: (Qi) asserts that given any two n-tuples $\mathbf{g} = [g_1, \ldots, g_n]$ and $\mathbf{h} = [h_1, \ldots, h_n]$, there are transformations α_1 and α_2, both of the type defined in condition (Pi), and indices $\varepsilon_1, \varepsilon_2 = \pm 1$ such that, for $j = 1, \ldots, n$, we have

$$\mathbf{hg} = \mathbf{g}^{\varepsilon_1 \alpha_1} \mathbf{h}^{\varepsilon_2 \alpha_2}.$$

Clearly (Qi) holds if and only if the appropriate subgroup of the diagonal group (as in Proposition 12) is stratifiable. In particular, $D(T, n)$ is stratifiable if and only if (Q3) holds for T and n.

Theorem 16 *If the diagonal group $D(T, n)$ is stratifiable for $n \geq 9$, then T is abelian.*

PROOF. Suppose that $gh \neq hg$. Consider the n-tuples $[g, g, 1, 1, 1, g, \ldots, g]$ and

$[1,1,h,h,h,h,\dots,h]$, where the last block contains $n-5$ symbols (and $n-5 \geq 4$). The product in reverse order is $(g,g,h,h,h,hg,\dots,hg)$. We have to show that no choice of α_i and ε_i is possible.

First we show that the α_i cannot involve τ. For τ would map the first tuple to one with four entries g^{-1} and $n-4$ entries 1, and the second to a tuple with three entries h^{-1} and $n-3$ entries 1. So if we used τ twice, the product would contain 1 at least twice. If we use it once, and the product of the two n-tuples doesn't contain the identity, then the numbers of elements of the three types are either $2,2,n-4$ or $3,n-3$, neither of which matches $2,3,n-5$.

So we can assume that α_1 and α_2 are automorphisms, and that $g^{\varepsilon_1\alpha_1} = g$, $h^{\varepsilon_2\alpha_2} = h$, and $g^{\varepsilon_1\alpha_1}h^{\varepsilon_2\alpha_2} = hg$, from which we conclude that $gh = hg$, contrary to assumption. $\square$

Possibly this bound can be reduced to 8. Since $D(Q_8,7)$ is generously transitive, it cannot be reduced to 7. Similar arguments show that the corresponding bounds for (Q1) and (Q2) are 3 and 6 respectively.

Question 3 *Does a similar result hold with "AS-friendly" or "AS-free" replacing "generously transitive" or "stratifiable" (assuming T non-abelian simple in the last case)?*

7 Primitive diagonal groups with few factors

We conclude with the promised construction of association schemes for diagonal groups with fewer than four factors in the socle.

Theorem 17 *Let T be a non-abelian simple group, and suppose that $n \leq 2$. Then $D(T,n)$ is not AS-free.*

PROOF. For $n = 1$, we have $\Omega = T$, and the permutation (d) is simply $t \mapsto t^{-1}$, so $D(1,T)$ is generously transitive. Since it is clearly not 2-transitive, it preserves a non-trivial association scheme.

We can say more about the minimal association scheme in this case. Using the identification of Ω with T, we see that the associate classes with respect to the identity have the form $O \cup O^{-1}$, where O is an orbit of $\mathrm{Aut}(T)$ on T.

For $n = 2$, we have, $\Omega = T^2$. The group T^3 preserves three congruences on Ω, the orbits of the three direct factors:

$[t_1,t_2] \sim_1 [u_1,u_2] \Leftrightarrow t_1^{-1}t_2 = u_1^{-1}u_2;$
$[t_1,t_2] \sim_2 [u_1,u_2] \Leftrightarrow t_1 = u_1;$
$[t_1,t_2] \sim_3 [u_1,u_2] \Leftrightarrow t_2 = u_2.$

These three congruences are preserved by permutations of types (a) and (b), and are permuted among themselves by types (c) and (d). (All is clear for types (a) and (b), on noting that $t_1^{-1}t_2 = u_1^{-1}u_2$ implies $x_1^{-1}t_1^{-1}t_2x_2 = x_1^{-1}u_1^{-1}u_2x_2$. Swapping the coordinates fixes $\sim_1$ and interchanges $\sim_2$ and $\sim_3$, while the permutation (d) swaps $\sim_1$ and $\sim_3$ and fixes $\sim_2$.

Now the three classes

$$C_1 = \{(\alpha,\alpha) : \alpha \in \Omega\},$$
$$C_2 = \{(\alpha,\beta) : \alpha \neq \beta, \alpha \sim_i \beta \text{ for some } i \in \{1,2,3\}\},$$
$$C_3 = \Omega^2 \setminus (C_1 \cup C_2),$$

form an association scheme, preserved by the diagonal group $D(T,2)$. In fact this association scheme is of *Latin square type*. The array with rows and columns indexed by T, with (t_1,t_2) entry $t_1^{-1}t_2$, is a Latin square (it is the multiplication table of T, slightly twisted); and C_1 consists of all pairs of cells of the array which lie in the same row, or in the same column, or carry the same entry. Thus C_1 is the edge set of a strongly regular *Latin square graph*. $\square$

We do not know whether diagonal groups with $n = 2$ are AS-friendly. The association scheme just constructed is not necessarily the minimal one. Indeed, in the case $T = A_5$, for $n = 2$ the property (P1) defined earlier can be shown to hold, and so $D(A_5,2)$ is generously transitive.

Note that the smallest diagonal group for which we have not been able to decide the AS-freeness is $D(A_5,3)$, with degree $60^3 = 216000$ (rather large for computation!)

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