Decompositions of complete multipartite graphs

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Abstract

This paper answers a recent question of Dobson and Marušič by partitioning the edge set of a complete bipartite graph into two parts, both of which are edge sets of arc-transitive graphs, one primitive and the other imprimitive. The first member of the infinite family is the one constructed by Dobson and Marušič.

Key words: arc-transitive graph, complete multipartite graph, primitive
2000 MSC: 05C25

In a recent paper [1], Dobson and Marušič ask the following question:

> Is there an infinite family of arc-transitive graphs with imprimitive automorphism group each of which can be decomposed into two (or more) arc-transitive graphs, (at least) one of which has a primitive automorphism group, while (at least) one of which has an imprimitive automorphism group?

The purpose of this note is to generalise the authors' own construction to give a family of such examples.

Construction. Let A be a m-dimensional affine space over the field $\mathrm{GF}(2)$, for $m \geq 3$. The vertex set of the graph will be the set of 2-element subsets of A; in other words, the affine lines. Now we partition the complete graph on V into three graphs X_0, X_1 and X_2 as follows: two affine lines are adjacent in X_0 if they are parallel; in X_1 if they intersect; and in X_2 if they are skew.

Now X_0 consists of $2^m - 1$ complete graphs each of size 2^{m-1} (corresponding to the parallel classes of A); it and its complement are arc-transitive and imprimitive, the automorphism group being just the wreath product of symmetric groups.

The graph X_1 is the "triangular graph" $T(2^m)$ [2], the line graph of K_{2^m}; it is arc-transitive and primitive, its automorphism group being the symmetric group $\mathrm{Sym}(2^m)$ (not depending on the affine structure).

The arc-transitivity of X_2 follows from the fact that the affine group $G = \mathrm{AGL}(m, 2)$ is transitive on affine-independent 4-tuples. We will show that this group is the full automorphism group of X_2, which is thus imprimitive.

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Consider two vertices which are not adjacent in X_2, and count their common neighbours. It is straightforward to show that, if the two vertices are intersecting pairs (that is, adjacent in X_1), they have $(2^m-4)(2^m-5)/2$ common neighbours, whereas if they are parallel (that is, adjacent in X_0), they have $(2^m-4)(2^m-6)/2$ common neighbours.

Thus, the graph structure of X_2 distinguishes the two types of non-edges (corresponding to edges of X_0 and X_1), and so $\mathrm{Aut}(X_2)$ preserves the parallelism relation and so is imprimitive. Moreover, from the graph structure, we can recover both the points of the affine space (the cliques of size 2^m-1 in the graph X_1) and the parallelism; so $\mathrm{Aut}(X_2) \leq \mathrm{AGL}(m,2)$, whence equality holds.

An alternative argument avoids this counting. By transitivity, the number of common neighbours of two non-adjacent vertices in X_2 depends only on which of X_0 and X_1 contains the pair of vertices. If these numbers are different, then the argument of the preceding paragraph applies. But if they are the same, then we have an *amorphous cellular ring* of rank 4, a partition of the edges of the complete graph into three strongly regular graphs. Ivanov [3] showed that this can only occur if the number of vertices is a square, which is false in our situation.

The case $m=3$ is the example given in the cited paper. The authors note that there is a cyclic automorphism having 4 cycles of length 7. This also generalises: in the examples presented here, a Singer cycle in the general linear group, acting on the affine space, has 2^{m-1} cycles of length 2^m-1 on vertices (affine lines).

References

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