Problems from CGCS
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Abstract

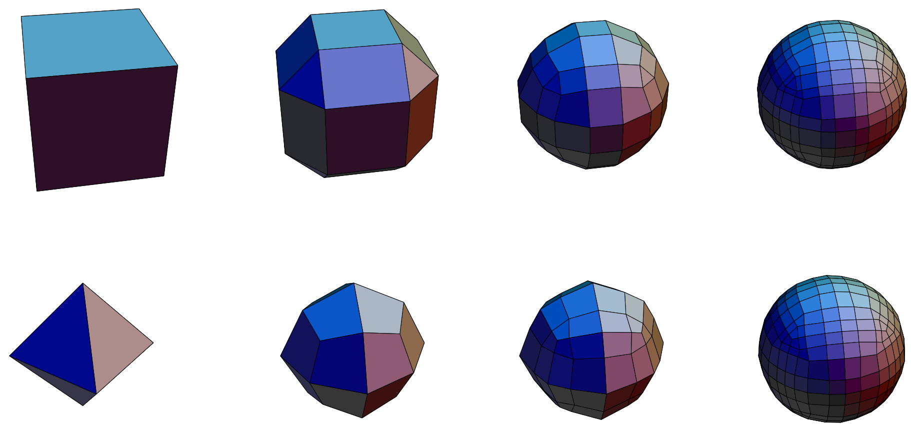
Most of these problems were presented at the conference on Combinatorics, Geometry and Computer Science, held at CIRM, Luminy, 2-4 May 2007. I have added one problem on a similar theme after the meeting.

The problems have been arranged so that those with similar themes occur together as far as possible.

1. Nearly round polytopes Communicated by Komei Fukuda.

We call a polytope *centered* if it contains the origin in its relative interior. Nesterov [17] has recently proved that the Minkowski sum of a centered full-dimensional polytope (and more generally a centered full-dimensional compact convex body) with a properly scaled copy of its dual gives a set whose asphericity is at most the square root of that of the initial polytope. The *asphericity* of a set is here defined as the ratio of the diameter of its smallest enclosing ball to that of its largest enclosed ball. Thus, summing a polytope with its own dual has a strong *rounding effect.* For this reason, the Minkowski sum $P + \alpha P^*$, will be called a *Nesterov rounding* of a polytope

P for any positive scalar α. The following example shows repeated Nesterov roundings of a cube.



One natural question is whether the Nesterov rounding gives a better rounding effect than fullerenes in dimension 3, where a fullerene F_n is a simple 3-polytope with only pentagonal and hexagonal facets. For example, does the asphericity of the 3-cube decrease faster than that of the optimal F_n as a function of the number of facets? What about the same question in terms of the *isoperimetric quotient* (IQ), see [10]? (The IQ of a polytope of a solid body in $\mathbb{R}^3$ is defined as $36\pi V^2/S^3$; it is at most 1, with equality only for the sphere.)

Finally, what is the relation between the asphericity and the IQ?

2. Sets of vectors with few inner products Communicated by Eiichi Bannai and Etsuko Bannai.

Let $X \subset \mathbb{R}^n$ and s a positive integer. Suppose that X is an s-inner product set, that is, the number of distinct inner products of distinct vectors in X is s.

Deza and Frankl [11, Theorem 1.4] proved that

$$|X| \leq \binom{n+s}{s}.$$

(a) Find examples of such sets X with $|X| = \binom{n+s}{s}$ for $n \geq 2$ and $s \geq 2$. (Furthermore, classify them if possible.)

(b) Deza and Frankl state:

> As pointed out by the referee, Theorem 1.4 can be deduced also using the approach of Koornwinder.

Is this really so?

Note: Koornwinder [15] gave a short proof of the result of Delsarte, Goethals and Seidel [9] that, if X is an s-distance set in the sphere S^{n-1}, then

$$|X| \leq \binom{n-1+s}{s} + \binom{n-2+s}{s-1}.$$

3. A finiteness condition Communicated by A. A. Ivanov.

This problem is based on Conway's proof [7] of the finiteness of the automorphism group of the Griess algebra. (This automorpism group is the *Monster*, the largest sporadic finite simple group.)

Let $V = \mathbb{C}^n$, with the standard positive definite inner product $\langle\ ,\ \rangle$. Let $*$ be a commutative but not necessarily associative product on V, that is, a symmetric bilinear map $V \times V \to V$. Assume that the associated trilinear form $(u, v, w) \mapsto \langle u * v, w\rangle$ on V is totally symmetric.

Let $\mathcal{F}$ be a set of idempotents (elements $v \in V$ such that $v * v = v$. Thus, the map $\tau_v : u \mapsto s * v$ has an eigenvalue 1). Assume that there is a constant λ with $0 \leq \lambda \leq \frac{1}{2}$ such that any eigenvector of τ_v which is not a multiple of v has eigenvalue of norm at most λ.

Problem: Find an explicit upper bound on $|\mathcal{F}|$ in terms of n and λ. (The finiteness of $\mathcal{F}$ follows from Conway's argument.)

4. Two problems on unimodular systems of vectors Contributed by V. Grishukhin.

A spanning set of vectors $U \subseteq \mathbb{R}^n$ is called *unimodular* if, for any subset $B \subseteq U$ which is a basis for $\mathbb{R}^n$, the coordinates of all vectors in U with respect to B are integers.

Problem 1 The Veronese map $u \mapsto uu^\top$, for $u \in \mathbb{R}^n$, takes a vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ to a vector

$$\tau(u) = (u_i u_j : 1 \leq i \leq j \leq n) \in \mathbb{R}^{n(n+1)/2}.$$

Give a direct proof of the fact that, if U is unimodular, then $\tau(U)$ is linearly independent. (There is an indirect proof of this assertion.)

Problem 2 Let $U \subseteq \mathbb{R}^n$ be unimodular. Under what conditions does there exist a basis B for $\mathbb{R}^n$ such that all vectors of U have coordinates 0 or 1 relative to this basis?

This is true for graphical unimodular systems and for the exceptional system R_{10} of ten vectors in $\mathbb{R}^5$ having either of the two representations shown:

$$R_{10} \approx \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

What about cographic unimodular systems?

5. Covering and packing planar graphs with balls Communicated by Victor Chepoi.

Let $\rho(G)$ be the minimum number of balls of radius R covering a graph $G = (V, E)$ and let $\gamma(G)$ be the maximum number of pairwise disjoint balls of radius R of G.

Question 1: Is it true that $\rho(G) \leq c\gamma(G)$ for some universal constant and all planar graphs G?

For $\gamma(G) = 1$ this question was raised in [13] and recently solved in [5]. A positive answer to Question 1 would be obtained by solving the following question.

Question 2: Is it true that any planar graph G contains a ball of radius $2R$ which can be covered with a constant number of balls of radius R (the *weak doubling property*)?

6. Finding a properly edge-coloured cycle Contributed by Y. Manoussakis.

Given a c-edge-coloured complete graph on n vertices, for $c \geq 3$, is there a polynomial algorithm to decide whether there is a properly edge-coloured

Hamiltonian cycle (one in which adjacent edges on the cycle have different colours), or to find one if it exists? (This problem was first posed in [1].)

For two colours, this problem is known to be polynomial [3].

7. Generalised Sudoku Contributed by Peter J. Cameron.

A *gerechte design* of order n consisting of an $n \times n$ grid whose n^2 cells are partitioned in to n 'regions' $S_1, \ldots, S_n$, each containing n cells, and an assignment of a number from the set $\{1, \ldots, n\}$ to each cell, such that each number occurs once in each row, column or region of the grid.

Gerechte designs were first introduced for agricultural statistics [2]. The case where $n = 9$ and the regions are 3×3 subsquares is currently popular under the name *Sudoku*.

Another example: given a Latin square L of order n, let S_i be the set of positions where symbol i occurs in L, for $i = 1, \ldots, n$. A gerechte design for this partition is precisely a Latin square orthogonal to L.

Problem: Given a partitioned grid, what is the computational complexity of finding a gerechte design for it (or of deciding whether one exists)?

Colbourn [6] showed that the decision problem for completing a partial Latin square is NP-complete; but the present problem is a bit different.

8. Incidence matrix of trees and forests Contributed by Nicolas M. Thiery.

Consider acyclic symmetric simple graphs with n labelled nodes $1, \ldots, n$ (also known as free, or labelled *forests*). Let $F_{n,d}$ be the set of subsets of those graphs with d edges. In particular, $F_{n,n-1}$ is the set of free *trees*, and has size n^{n-2} by Cayley's Theorem.

Consider the incidence matrix $M_n = (a_{t,f})$ whose rows are indexed by trees t in $F_{n,n-1}$, and columns by forests f in $F_{n,n-2}$ (these are sometimes called *thickets*), such that

$$a_{t,f} = \begin{cases} 1 & \text{if } f \text{ is a subgraph of } t, \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture: M_n is of maximal rank (that is, its columns are linearly independent).

This has been proved by hand and computer for $n \leq 7$; a lower bound on the rank is known for $n > 7$ [20].

There is a variant of the conjecture for unlabelled forests: instead, we take $a_{t,f}$ to be the number of embeddings of f into t. This form of the conjecture has been checked by computer for all $n \leq 19$.

This problem is related to a conjecture of Kocay [14] on the reconstruction of the number of spanning subtrees of a given type in a graph. See Bondy's survey [4] for more information about reconstruction.

9. One missing intersection Communicated by Mike Newman.

Let $n = 4m$, and let $\mathcal{S}$ be a set of subsets of $[n] := \{1, \ldots, n\}$, such that for any $A, B \in S$, $|A\Delta B| \neq \frac{n}{2}$. What is the maximum size of such a set $\mathcal{S}$?

The conjecture is that the maximum should be achieved as follows. For a set of sets $\mathcal{A}$, let $\overline{\mathcal{A}} = \{\overline{A} : A \in \mathcal{A}\}$, and let $\mathcal{A}\Delta F = \{A\Delta F : A \in \mathcal{A}\}$. Define

$$\mathcal{R} = \{A \subset [n] \ : \ |A| < m, \ |A| \not\equiv m \pmod 2\}$$

Then the largest possible $\mathcal{S}$ should be achieved by

$$(\mathcal{R} \cup \overline{\mathcal{R}}) \cup (\mathcal{R} \cup \overline{\mathcal{R}})\Delta\{1\}$$

(Any set of odd cardinality would do in place of $\{1\}$).

Let G be the graph whose vertices are the subsets of $[n]$ and where two subsets are adjacent if their symmetric difference has size $\frac{n}{2}$. We want the independence number of G. It is not hard to see that G has two components (the vertices of even and odd weight), and these are isomorphic. Furthermore, each component is a lexicographic product of the form $H[\overline{K_2}]$. So $\mathcal{S}$ is a maximum independent set in G if and only if $\mathcal{R}$ is a maximum independent set in H if and only if $\mathcal{R} \cup \overline{\mathcal{R}}$ is a maximum independent set in $H[\overline{K_2}]$.

See [18] for more background. If true, this conjecture would imply an asymptotic result of Frankl and Rödl [12] and improve the upper bound on $\chi(G)$. It has been verified up to $n = 16$ using Schrijver's semidefinite programming based on the Terwilliger algebra [19, 8]. Furthermore, for $n = 8$, the above construction is the only maximum independent set, up to graph isomorphism (determined by Gordon Royle).

10. A generating function question Contributed by Patrick Solé.

Let

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Is there a generating function in closed form for the sequence with nth term

$$\log(H_n)\exp(H_n)?$$

This is motivated by a result of Lagarias [16], who showed that the statement

$$\sigma(n) \leq H_n + \log(H_n)\exp(H_n) \text{ for all } n \geq 1$$

is equivalent to the Riemann Hypothesis, where $\sigma(n)$ is the sum of the (positive) divisors of n.

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