Cores of symmetric graphs*

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Abstract

The core of a graph Γ is the smallest graph Δ which is homomorphically equivalent to Γ (that is, there exist homomorphisms in both directions). The core of Γ is unique up to isomorphism and is an induced subgraph of Γ.

We give a construction in some sense dual to the core. The *hull* of a graph Γ is a graph containing Γ as a spanning subgraph, admitting all the endomorphisms of Γ, and having as core a complete graph of the same order as the core of Γ. This construction is related to the notion of a synchronizing permutation group which arises in semigroup theory; we provide some more insight by characterizing these permutation groups in terms of graphs.

It is known that the core of a vertex-transitive graph is vertex-transitive. In some cases we can make stronger statements: for example, if Γ is a nonedge-transitive graph, we show that either the core of Γ is complete, or Γ is its own core.

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Rank 3 graphs are nonedge-transitive. We examine some families of these to decide which of the two alternatives for the core actually holds. We will see that this question is very difficult, being equivalent in some cases to unsolved questions in finite geometry (for example, about spreads, ovoids, and partitions into ovoids in polar spaces).

1 Introduction

In this paper, we speak of graphs and groups; a typical graph will be called Γ and group G. Groups (and semigroups) will act on the right, so that g maps v to vg. All graphs will be finite, undirected, and simple.

A *graph homomorphism* from Γ to Γ' is a map from the vertex set of Γ to that of Γ' which maps edges to edges. (There is no condition on non-edges.) Two graphs are *homomorphism-equivalent* if there are homomorphisms in both directions between them. This is an equivalence relation coarser than isomorphism. Each equivalence class contains a unique graph (up to isomorphism) with fewest vertices; such a graph is a *core*, or the *core* of any graph in the class. The core of a graph can be embedded as an induced subgraph, and there is a *retraction* from a graph to its core (a homomorphism which fixes its image pointwise). We will sometimes denote the core of Γ by $\mathrm{Core}(\Gamma)$. The core of a graph Γ is complete if and only if the clique number $\omega(\Gamma)$ (the size of the largest complete subgraph) and the chromatic number $\chi(\Gamma)$ (the smallest number of colours required for a proper colouring of the vertices) are equal. For a k-clique exists if and only if there is a homomorphism $K_k \to \Gamma$, and a k-colouring exists if and only if there is a homomorphism $\Gamma \to K_k$.

An *endomorphism* is a homomorphism from a graph to itself. The endomorphisms of a graph form a semigroup. An endomorphism of a finite graph is an automorphism if and only if it is one-to-one. (For a bijective endomorphism cannot decrease the number of edges, and cannot increase it either.) A graph is a core if and only if all its endomorphisms are automorphisms.

The *rank* of a transitive permutation group G on a set Ω is the number of orbits of G on Ω^2.

We refer to [6] for more information on permutation groups, and [10, 11] for graphs, homomorphisms and cores. The inspiration that led us to the results in Section 2 derives from the papers [2, 1], which describe the background in automata theory, and [17], with which this paper has a lot in common (although the aims are quite different). We are grateful to the authors for helpful comments. We have described the connection in Subsection 2.1.

2 Cores and hulls

Welzl [20] showed that the core of a vertex-transitive graph is vertex-transitive; see [10, Theorem 3.7]. More general statements are true; for example, the properties of edge-transitivity and nonedge-transitivity are inherited by cores as well.

In this section we show that, for nonedge-transitivity, much more is true: if Γ is nonedge-transitive, then either the core of Γ is complete, or Γ is a core. These results will be deduced from a much more general theorem which applies to all graphs.

Let Γ be a graph; let $G = \mathrm{Aut}(\Gamma)$ be its automorphism group, and $S = \mathrm{End}(\Gamma)$ its endomorphism semigroup. We define a graph Γ', which we will call the *hull* of Γ, as follows:

(a) Γ' has the same vertex set as Γ;

(b) two distinct vertices v, w are adjacent if Γ' if and only if there does *not* exist an element $f \in \mathrm{End}(\Gamma)$ with $vf = wf$.

The next result gives some properties of the hull.

Theorem 2.1 *Let Γ' be the hull of Γ.*

(a) *Γ is a spanning subgraph of Γ'.*

(b) $\mathrm{End}(\Gamma) \le \mathrm{End}(\Gamma')$ *and* $\mathrm{Aut}(\Gamma) \le \mathrm{Aut}(\Gamma')$.

(c) *If the core of Γ has k vertices, then the core of Γ' is the complete graph K_k on k vertices.*

Proof (a) Let v and w be adjacent in Γ, and $f \in S = \mathrm{End}(\Gamma)$. Then vf and wf are adjacent in Γ, so $vf \ne wf$. Thus v and w are adjacent in Γ', by definition.

(b) Let $f \in \mathrm{End}(\Gamma)$, and let v and w be adjacent in Γ'; we have to show that vf and wf are adjacent in Γ'. But, for any $h \in S$, $(vf)h = v(fh)$ and $(wf)h = w(fh)$ are distinct, by definition of the hull. So the assertion is proved.

Now an endomorphism of a finite graph is an automorphism if and only if it is one-to-one, so the inclusion of the automorphism groups follows.

(c) Let K be a core of Γ. There is an endomorphism h of Γ with image K, but no endomorphism with smaller image; so no endomorphism f of Γ can map two vertices of K to the same vertex, else hf has smaller image than h. So by definition of the hull, K induces a complete graph in Γ', with (say) k vertices. Then the map h is a proper k-colouring of Γ', so the core of Γ' is K_k.

In particular, Γ is a core if and only if its hull is complete. The "dual" statement is not true: if Γ is equal to its hull, then its core is complete, but the converse is not true. For example, the hull of a path of length 3 is a 4-cycle (note the extra symmetry), but its core is K_2 since it is bipartite.

We give a couple of corollaries concerning symmetric graphs. A graph is *nonedge-transitive* if its automorphism group is transitive on unordered pairs of nonadjacent vertices.

Corollary 2.2 *Let Γ be a nonedge-transitive graph. Then either the core of Γ is a complete graph, or Γ is a core.*

Proof Let Γ have n vertices, and let Γ' be the hull of Γ. By Theorem 2.1(a) and (b), Γ' is obtained from Γ by converting the nonedges in some $\mathrm{Aut}(\Gamma)$-orbits to edges. But there is only one such orbit. So either $\Gamma' = \Gamma$ (in which case the core of Γ is complete), or Γ' is complete (in which case its core has n vertices, so the same is true for Γ by Theorem 2.1(c), so that Γ is a core).

The second corollary is known [10, Theorem 3.9], but our proof is quite straightforward.

Corollary 2.3 *Let Γ be a vertex-transitive graph with n vertices, whose core has k vertices. Then k divides n.*

Proof We may replace Γ with its hull without changing the orders of the graph or its core or the fact that the graph is vertex-transitive. So we may assume that the core of Γ is complete. Thus the clique number and chromatic number of Γ are both equal to k. Choose a k-clique K, and a colouring with colour classes $C_1, \ldots, C_k$. We are going to show that $|C_i| = n/k$, from which the result will follow. Let $G = \mathrm{Aut}(\Gamma)$.

Let $\mathcal{B}$ be the set of images of K under G. If $|\mathcal{B}| = b$, then vertex-transitivity shows that every vertex lies in a constant number r of members of $\mathcal{B}$, where $nr = bk$.

Choose a colour class C_i, and count choices of (v, B), where $v \in C_i$, $B \in \mathcal{B}$, and $v \in B$. Clearly there are $|C_i|r$ such choices. But each element of $\mathcal{B}$ is a k-clique, so meets C_i in a vertex, so the number of choices is $|\mathcal{B}| = b$. So

$$|C_i| = b/r = n/k,$$

and we are done.

Remark We see from the proof that, if the core of a vertex-transitive graph Γ is complete, then the product of the clique number $\omega(\Gamma)$ and coclique number $\alpha(\Gamma)$ of Γ is equal to the number n of vertices. Now it is well-known (and proved by almost the same argument) that, in an arbitrary vertex-transitive graph, we have $\omega(\Gamma)\alpha(\Gamma) \leq n$; so equality is a necessary condition for the core to be complete. Since the product is not changed when Γ is replaced by its complement, this is also a necessary condition for the core of $\overline{\Gamma}$ to be complete. (The corollary and remark are inspired by a result of Neumann, see [17, Theorem 2.1]; see also Lemma 2.6 in that paper.)

2.1 Synchronizing permutation groups

Although the above material is self-contained, the inspiration for it came from results on synchronizing permutation groups by Arnold and Steinberg [2] and Araújo [1]. In fact there is a close connection between this theory and our results. The concepts arise from automata theory, where the existence of a reset word and a conjectured bound on its length if it exists (the *Černý problem*) are among the oldest problems in the theory. We are grateful to João Araújo, Peter Neumann and Ben Steinberg for providing us with this information. Indeed, the relation between parts of this paper and [1, 17, 2] is very close, although the context is different, since these authors consider permutation groups rather than graphs.

A permutation group G on a set V is called *synchronizing* if the following holds: for any map $f : V \to V$ which is not a permutation, the semigroup generated by G and f contains a constant function. It is known that a synchronizing permutation group is primitive, but not every primitive group is synchronizing.

Theorem 2.4 *(a) Let* Γ *be a non-null graph which is not a core. Then* $\mathrm{Aut}(\Gamma)$ *is not synchronizing.*

(b) Conversely, let G *be a permutation group which is not synchronizing. Then there is a non-null graph* Γ*, which is not a core, such that* $G \leq \mathrm{Aut}(\Gamma)$.

Proof (a) Let $S = \mathrm{End}(\Gamma)$. Then S contains $\mathrm{Aut}(\Gamma)$, and contains a function which is not a permutation (since Γ is not a core), but does not contain a constant function (since Γ is non-null). So $\mathrm{Aut}(\Gamma)$ is not synchronizing.

(b) Suppose that G is not synchronizing, and choose a semigroup S containing G and containing a function which is not a permutation, but containing no constant function. Now we follow the construction of the hull: we let Γ be the graph on the vertex set V, in which v and w are joined if there is no element $f \in S$ satisfying $vf = wf$. As in the proof of Theorem 2.1, Γ is non-null, and $S \leq \mathrm{End}(\Gamma)$ (so that $G \leq \mathrm{Aut}(\Gamma)$ and Γ is not a core).

It follows that a rank 3 permutation group is synchromizing if and only if both of its orbital graphs are cores. Our results later in the paper will show that even for rank 3 groups G, it is difficult to decide whether G is synchronizing or not.

Another consequence of the theorem is the fact that a permutation group G is synchronizing if and only if its 2-closure is. (The 2-*closure* of G consists of all permutations which preserve every G-orbit on ordered pairs.)

3 Rank 3 graphs

A *rank* 3 *graph* is a graph whose automorphism group is transitive on vertices, ordered edges, and ordered non-edges. In other words, it is a (non-diagonal) orbital graph of a rank 3 permutation group of even order. (The study of such graphs was begun by D. G. Higman [12].) All such permutation groups have been determined [13, 15, 14], so in principle all such graphs are known. We will see that there are things we don't know about these graphs.

By Corollary 2.2, if Γ is a rank 3 graph, then either the core of Γ is complete, or Γ is a core. In this section, we examine some families of rank 3 graphs to see which alternative actually holds. We will see that this leads to some difficult questions in finite geometry. We consider only primitive rank 3 graphs, since an imprimitive rank 3 graph is either a disjoint union of complete graphs, or a complete multipartite graph, and its core is complete.

The remarks in the previous section show that, if Γ is a rank 3 graph on n vertices which satisfies $\omega(\Gamma)\alpha(\Gamma) < n$, or for which $\omega(\Gamma)$ does not divide n, then both Γ and its complement $\overline{\Gamma}$ are cores. Indeed, Γ is a core if and only if $\omega(\Gamma) < \chi(\Gamma)$. While both clique number and chromatic number are NP-hard to compute, in practice the former is much easier, using tools such as GRAPE [18].

3.1 Square lattice graphs

The *square lattice graph* $L_2(n)$ has as vertices the ordered pairs (i, j), with $1 \leq i, j \leq n$; vertices (i, j) and (i', j') are adjacent if $i = i'$ or $j = j'$ (but not both). It is a rank 3 graph, admitting the wreath product S_n wr S_2 in its product action. It can also be regarded as the line graph of the complete bipartite graph $K_{n,n}$.

Proposition 3.1 *The cores of $L_2(n)$ and its complement are both K_n.*

Proof In $L_2(n)$, a row or column of the square grid is a clique, and a Latin square of order n gives an n-colouring. In the complement, the set $\{(i, ig) : i \in \{1, \ldots, n\}\}$, for any permutation g, is an n-clique, while the rows of the grid give a partition into cocliques.

This result is closely connected with [17, Example 3.4].

3.2 Triangular graphs

The *triangular graph* $T(n)$ is the graph whose vertices are the 2-element subsets of $\{1, \ldots, n\}$, two vertices adjacent if the sets have non-empty intersection. It is a rank 3 graph, admitting the symmetric group S_n. It can also be described as the line graph of the complete graph K_n.

Proposition 3.2 *(a) For $n \geq 5$, the core of $T(n)$ is K_{n-1} if n is even, and $T(n)$ if n is odd.*

(b) For $n \geq 5$, $\overline{T(n)}$ is a core.

Proof (a) The clique number of $T(n)$ is $n - 1$, a maximal clique consisting of all the 2-subsets containing a fixed element of $\{1, \ldots, n\}$. Now an $(n-1)$-vertex colouring of $T(n)$ is an $(n-1)$-edge colouring of K_n; these are well known to exist if and only if n is even.

(b) The clique number of $\overline{T(n)}$ is $\lfloor n/2 \rfloor$, a maximal clique consisting of pairwise disjoint 2-sets. A theorem of Lovász [16] shows that the chromatic number of this graph is $n-2$, which is greater than $\lfloor n/2 \rfloor$ for $n \geq 5$. (In fact, we do not need to use Lovász's Theorem here. The only cocliques of size $n-1$ consist of all pairs containing a fixed element of $\{1, \ldots, n\}$, and there cannot be a partition into such sets, since any two of them intersect in one vertex.)

This result is closely connected with [17, Example 3.7].

3.3 Paley graphs

Let q be a prime power congruent to 1 mod 4. The *Paley graph* $P(q)$ has vertex set the finite field $\mathrm{GF}(q)$, two vertices joined if their difference is a non-zero square. It is a rank 3 graph, admitting the group of additions and multiplications by non-zero squares. Multiplication by a non-square induces an isomorphism from the graph to its complement.

Proposition 3.3 *Let q be a prime power congruent to* 1 *mod* 4.

(a) If q is not a square, then $P(q)$ is a core.

(b) If $q = r^2$, then the core of $P(q)$ is K_r.

Proof (a) Since $\Gamma = P(q)$ is self-complementary, its clique and coclique numbers are equal; if q is not a square, their product cannot be q, so Γ is a core.

(b) Let $V = \mathrm{GF}(q)$ and $W = \mathrm{GF}(r)$. Then every element of W is a square in V, so W is a clique. If a is a non-square, then Wa is a coclique, as are its translates $Wa + b$; there are r such translates, giving an r-colouring of Γ.

3.4 Line graphs of projective spaces

The line graph of the projective space $\mathrm{PG}(n, q)$ has as vertices the lines of the space, two vertices joined if the lines intersect. It is a rank 3 graph on $(q^{n+1}-1)(q^n-1)/(q^2-1)(q-1)$ vertices, admitting the group $\mathrm{PGL}(n+1, q)$.

Proposition 3.4 *Let Γ be the line graph of $\mathrm{PG}(n, q)$, where $n \geq 3$.*

(a) The core of Γ is the complete graph of order $(q^n - 1)/(q - 1)$ if there is a parallelism of the lines of $\mathrm{PG}(n, q)$; otherwise Γ is a core.

(b) $\overline{\Gamma}$ is a core.

Proof (a) For $n > 3$, a clique of maximum size consists of all lines through a point, and contains $(q^n - 1)/(q - 1)$ vertices. (If $n = 3$, theare are other cliques of maximum size, cnsisting of all lines in a plane.) A coclique has size at most $(q^{n+1} - 1)/(q^2 - 1)$, with strict inequality unless $q^2 - 1$ divides $q^{n+1} - 1$, that is, 2 divides $n + 1$. So, if n is even, then Γ is a core. If n is odd, a maximum coclique is a spread, and a colouring with $(q^n - 1)/(q - 1)$ such cocliques is a partition into spreads, that is, a parallelism.

(b) If the core of $\overline{\Gamma}$ is a clique, then the vertices can be partitioned into cocliques of size $(q^n - 1)/(q - 1)$. For $n > 3$, such a coclique consist of the lines through a point, and any two of them intersect, so no partition is possible. For $n = 3$, there is one further type, but we can have at most two pairwise disjoint such cocliques, one of each type. So $\overline{\Gamma}$ is a core.

A necessary condition for a parallelism of the lines of $\mathrm{PG}(n, q)$ is that n is odd (this is the necessary and sufficient condition for a spread, as we saw). It is conjectured that this is sufficient, but this has only been proved in special cases:

(a) for $n = 3$ (Denniston [7]);

(b) more generally, for $n = 2^d - 1$, $d \geq 2$ (Beutelspacher [4]);

(c) for $q = 2$ (Baker [3]).

The first open case is thus $\mathrm{PG}(5, 3)$, whose line graph has 11011 vertices.

3.5 Point graphs of polar spaces

A classical polar space is defined by a form of one of the following types on a finite-dimensional vector space: a non-degenerate alternating bilinear form; a non-degenerate Hermitian form; a non-singular quadratic form. The points are the 1-dimensional subspaces which are totally isotropic or totally singular with respect to the form. The graph of the polar space has the points as its vertices, two points being adjacent if they are orthogonal with respect to the form. See [5, 19] for further discussion.

Over a finite field, the graph of a classical polar space is a rank 3 graph, admitting the corresponding classical group, provided the Witt index of the form (the dimension of the largest totally isotropic or totally singular subspace) is at least 2, and excluding quadratic forms in odd dimension over fields of characteristic 2. We denote these by the notation standard in finite geometry. Here n is the projective dimension (one less than the vector space dimension), and q is the field order. By abuse of notation, we denote the graph by the same symbol as the polar space. So the examples are

(a) $W_n(q)$, for n odd and $n \geq 3$ (alternating form);

(b) $H_n(q)$, for $n \geq 3$ and q a square (Hermitian form);

(c) $Q_n^+(q)$ (for n odd, $n \geq 3$), $Q_n^-(q)$ (for n odd, $n \geq 5$), $Q_n(q)$ (for n even, $n \geq 4$, q odd).

We refer to [19] for further details.

The graph $Q_3^+(q)$ comes from the ruled quadric in projective 3-space and is isomorphic to $L_2(q+1)$, which we have already considered. So we may disregard this case.

In a polar space, a *generator* is a maximal totally isotropic or totally singular subspace (isomorphic to $\mathrm{PG}(r-1, q)$, where r is the Witt index). An *ovoid* is a set of points meeting every generator in one point, and a *spread* is set of generators which partition the point set.

The number of points is $(q^r - 1)(q^{r+e} + 1)/(q-1)$, where e is a parameter depending on the type of space. A generator has $(q^r - 1)/(q-1)$ points; so a spread must consist of $q^{r+e} + 1$ generators, and an ovoid must consist of $q^{r+e} + 1$ points.

Theorem 3.5 *Let Γ be the graph of a finite classical polar space S.*
(a) The core of Γ is complete if and only if S has a partition into ovoids.
(b) The core of $\overline{\Gamma}$ is complete if and only if S has an ovoid and a spread.

Proof It follows from the properties of sesquilinear forms that if a point is orthogonal to two points of a totally isotropic or totally singular line, then it is orthogonal to every point on a line. So a maximal clique is a subspace, hence a generator. A coclique is a 'partial ovoid', meeting any generator in at most one point. So the product of the clique number and the coclique number is equal to the number of points if and only if ovoids exist.

Now a colouring of Γ is a partition into 'partial ovoids', while a colouring of $\overline{\Gamma}$ is a partition into subsets of generators. From this, both parts of the theorem follow.

It is a topic of great interest in finite geometry to decide which polar spaces have ovoids, spreads, or partitions into ovoids. The complete answer is not known. We summarise here the main implications for our question of what is known, referring to the fairly recent survey [19] for more information. Of course, if the core is not complete, then the graph is a core. Let Γ be the point graph of a classical polar space.

$W_n(q)$: Core(Γ) and Core($\overline{\Gamma}$) are complete if $n = 3$ and q even, not if $n = 3$ and q is odd or if $n > 3$.

$H_n(q)$: Core($\overline{\Gamma}$) is not complete for any $n \geq 4$. Core(Γ) is not complete if n is even.

$Q_{2m}(q)$, $m \geq 3$: Core($\overline{\Gamma}$) is complete if $m = 3$ and q is a power of 3.

$Q^+_{2m-1}(q)$, $m \geq 3$: Core Γ is complete for $m = 3$ and in a few cases for larger m; Core($\overline{\Gamma}$) is not complete if m is odd or if $q = 2$ or $q = 3$.

$Q^-_{2m-1}(q)$, $m \geq 3$: neither Core(Γ) nor Core($\overline{\Gamma}$) is complete.

3.6 Small rank 3 graphs

Using GAP [9] and its share package GRAPE [18], it is easy to investigate small rank 3 graphs. We can take the list of primitive permutation groups of given degree, and select those which have rank 3 and even order and are 2-closed (that is, the full automorphism groups of their orbital graphs). Then for graphs on up to 100 vertices, we can compute their clique and independence numbers; those for which the product of these numbers is smaller than n are cores. If the product is equal to n, further investigation is required. We give a couple of examples.

(a) For the primitive group of degree 64, number 47 (which is isomorphic to $2^6 : 3S_6$), the clique number of the orbital graph of valency 18 is 4, and the independence number is 16. (The cliques are the lines of a generalized quadrangle.) The 64 points have the structure of a 3-dimensional affine space over GF(4); an independent set of size 16 is a 2-dimensional subspace, and its translates give a 4-colouring of the orbital graph. So the core of the orbital graph is K_4. By almost identical argument the core of its complement is K_{16}.

(b) The group $HS:2$ of degree 100 (primitive group number 4) has the famous Higman–Sims graph as an orbital graph. This graph has clique number 2 (it is triangle-free, since it is 2-arc transitive) and coclique number 22 (the neighbours of a vertex form a coclique of maximum size). So both the graph and its complement are cores.

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