A NOTE ON HIGHER-DIMENSIONAL MAGIC MATRICES

PETER J. CAMERON[†], CHRISTIAN KRATTENTHALER[‡], and THOMAS W. MÜLLER[†]

[†] School of Mathematical Sciences,
Queen Mary & Westfield College, University of London,
Mile End Road, London E1 4NS, United Kingdom.
WWW: `http://www.maths.qmw.ac.uk/~pjc/`
WWW: `http://www.maths.qmw.ac.uk/~twm/`

[‡] Fakultät für Mathematik, Universität Wien,
Nordbergstraße 15, A-1090 Vienna, Austria.
WWW: `http://www.mat.univie.ac.at/~kratt`

ABSTRACT. We provide exact and asymptotic formulae for the number of unrestricted, respectively indecomposable, d-dimensional matrices where the sum of all matrix entries with one coordinate fixed equals 2.

1. INTRODUCTION

We begin by recalling the notion of a *magic matrix*:[1] this is a square matrix $m = (m_{i,j})_{1\le i,j\le n}$ with non-negative integral entries such that all row and column sums are equal to the same non-negative integer. If this non-negative integer is s, then we call such a matrix *s-magic*. The enumeration of s-magic squares has a long history, going back at least to MacMahon [11, §404–419]. A good account of the enumerative theory of magic squares can be found in [14, Sec. 4.6], with many pointers to further literature. For more recent work, see for instance [4, 8].

Let $[n]$ denote the standard n-set $\{1, 2, \dots, n\}$. There are two obvious ways of generalizing s-magic matrices to higher dimensions:

(G1) *All line sums are equal.* Given a positive integer d, a d-dimensional matrix $m : [n]^d \to \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of non-negative integers) is called *s-magic* if

$$\sum_{\omega_i\in[n]} m(\omega_1, \omega_2, \dots, \omega_d) = s \tag{1.1}$$

for all fixed $\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_d \in [n]$, and all $i = 1, 2, \dots, d$.

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05A16 05A19 05B15.

Key words and phrases. higher-dimensional magic matrices, labelled combinatorial structures, multisort species, exponential principle.

[‡]Research partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S9607-N13, the latter in the framework of the National Research Network "Analytic Combinatorics and Probabilistic Number Theory".

[1]Strictly speaking, the correct term here would be "s-semi-magic," since we do not require diagonals to sum up to the same number as the rows and columns, see e.g. [4]. However, here and in what follows we prefer the term "magic" for the sake of brevity.

(G2) *All hyperplane sums are equal.* Given a positive integer d, a d-dimensional matrix $m : [n]^d \to \mathbb{N}_0$ is called *s-magic* if

$$\sum_{\omega_1,\dots,\omega_{i-1},\omega_{i+1},\dots,\omega_d \in [n]} m(\omega_1, \omega_2, \dots, \omega_d) = s \tag{1.2}$$

for all fixed $\omega_i \in [n]$, and all $i = 1, 2, \dots, d$.

Generalisation (G1) appears already in the literature, see e.g. [1, 4]. For $d = 3$ and $s = 1$, these objects are equivalent to Latin squares counted up to isotopy: the roles of rows, columns, and symbols of the corresponding Latin square are played by the first, second, and third coordinate, respectively, and the entry in position (ω_1, ω_2) of the Latin square is ω_3 if and only if $m(\omega_1, \omega_2, \omega_3) = 1$.

Rather surprisingly, we have not been able to locate Generalisation (G2) of magic squares to higher dimensions in the literature. The present note focusses on this second generalization. Hence, from now on, whenever we use the term "s-magic," this is understood in the sense of (G2).

Counting higher-dimensional magic matrices is made more difficult (than the already difficult case of 2-dimensional magic matrices) by the fact that the analogue of Birkhoff's Theorem (cf. [5] or [2, Corollary 8.40]; it says that any 2-dimensional s-magic matrix can be decomposed in a sum of permutation matrices, that is, 1-magic matrices) fails for them. For example, the 3-dimensional 2-magic matrix with ones in positions $(1, 1, 1)$, $(1, 2, 3)$, $(2, 1, 2)$, $(2, 2, 1)$, $(3, 3, 2)$ and $(3, 3, 3)$ is not the sum of two 1-magic matrices.

As we demonstrate in this note, it is however possible to count the 2-magic matrices of any dimension. Our first result is a recurrence relation for the number $u_n(d)$ of indecomposable d-dimensional 2-magic matrices (see Corollary 3 in Section 4). This recurrence is used in Proposition 4 to derive, for fixed $d \geq 3$, an asymptotic formula for $u_n(d)$. In order to go from indecomposable matrices to unrestricted ones, we observe that the d-dimensional 2-magic matrices may be viewed as a d-sort species in the sense of Joyal [10] which obeys the (d-sort) exponential principle. Let $w_n(d)$ denote the number of *all* d-dimensional 2-magic matrices. The exponential principle can then be applied to relate the numbers $w_n(d)$ to the numbers $u_n(d)$, see (3.5) (for $d = 2$) and (6.1) (for $d \geq 2$). This relation is used in Theorem 5 to find, for fixed $d \geq 3$, an asymptotic estimate for the numbers $w_n(d)$ as well. Exact and asymptotic formulae for $u_n(d)$ and $w_n(d)$ for $d = 2$ are presented in Section 3. We remark in passing that a simple counting argument shows that the obvious interpretation of the matrices in Generalisation (G1) as a d-sort species does *not* satisfy the exponential principle, not even under the — in a sense — minimal axiomatics of [7].

2. Indecomposable 2-magic matrices and fixed-point-free involutions

A d-dimensional matrix $m : [n]^d \to \mathbb{N}_0$ is called *decomposable*, if there exist non-empty subsets $B_1^{(1)}, B_2^{(1)}, B_1^{(2)}, B_2^{(2)}, \dots, B_1^{(d)}, B_2^{(d)}$ of $[n]$ with

$$B_1^{(1)} \amalg B_2^{(1)} = B_1^{(2)} \amalg B_2^{(2)} = \cdots = B_1^{(d)} \amalg B_2^{(d)} = [n]$$

($\amalg$ denoting disjoint union) and

$$|B_1^{(1)}| = |B_1^{(2)}| = \cdots = |B_1^{(d)}|,$$

such that $m(\omega_1, \omega_2, \dots, \omega_d) \neq 0$ only if either

$$(\omega_1, \omega_2, \dots, \omega_d) \in B_1^{(1)} \times B_1^{(2)} \times \cdots \times B_1^{(d)}$$

or

$$(\omega_1, \omega_2, \dots, \omega_d) \in B_2^{(1)} \times B_2^{(2)} \times \cdots \times B_2^{(d)},$$

otherwise it is called *indecomposable.*[2] (In less formal language: there exist reorderings of the lines of the matrix such that m attains a block form.) The integer n is called the *size* of m.

Let $u_n(d)$ denote the number of indecomposable d-dimensional 2-magic matrices of size n. Note that an indecomposable 2-magic matrix with an entry 2 has size 1. So it is enough to consider zero-one matrices.

The purpose of this section is to relate the numbers $u_n(d)$ to another sequence of numbers $v_n(d)$ counting certain tuples of fixed-free involutions on a set with $2n$ elements. More precisely, let

$$t_1 = (1, 2)(3, 4) \cdots (2n-1, 2n), \tag{2.1}$$

be the standard fixed-point-free involution on the set $[2n]$. Then we define $v_n(d)$ to be the number of choices of $d-1$ fixed-point-free involutions $t_2, \dots, t_d$ on $[2n]$ such that the group $G = \langle t_1, t_2, \dots, t_d \rangle$ generated by $t_1, t_2, \dots, t_d$ is transitive. Then we have the following relation.

Lemma 1. *For all integers $n, d > 1$, we have*

$$u_n(d) = 2^{-n}(n!)^{d-1} v_n(d). \tag{2.2}$$

Proof. Let m be an indecomposable d-dimensional 2-magic matrix of size n, where $n > 1$. Then m is a zero-one matrix, and it contains $2n$ entries equal to 1, the rest being zero. Number the positions of the 1's in m from 1 to $2n$ in such a way that the positions with first coordinate j are numbers $2j-1$ and $2j$ for $j = 1, \dots, n$. (There are 2^n ways to do this.) Then, for $i = 1, \dots, d$, let t_i be the fixed-point-free involution whose cycles are the pairs of numbers in $\{1, \dots, 2n\}$ indexing positions of 1's with the same i-th coordinate. Note that t_1 is the involution defined in (2.1). It is straightforward to show that indecomposability of m implies transitivity of the group generated by these involutions. So each matrix gives rise to 2^n such d-tuples of involutions. Thus, the number of pairs consisting of a matrix and a corresponding sequence of permutations is $2^n\, u_n(d)$.

For instance, the example of a matrix failing the analogue of Birkhoff's Theorem given in the Introduction, with the entries numbered in the order given, produces the three permutations $(1, 2)(3, 4)(5, 6)$, $(1, 3)(2, 4)(5, 6)$ and $(1, 4)(2, 6)(3, 5)$.

Conversely, let $t_1, \dots, t_d$ be fixed-point-free involutions on the set $\{1, \dots, 2n\}$ which generate a transitive group, where t_1 is the standard involution defined in (2.1). Number the cycles of each t_i from 1 to n such that the cycle $(2j-1, 2j)$ of t_1 has number j. (There are $(n!)^{d-1}$ such numberings.) Now construct a d-dimensional matrix m as follows: for $k = 1, \dots, 2n$, if k lies in cycle number ω_i of t_i, then $m(\omega_1, \omega_2, \dots, \omega_d) = 1$; all other entries are zero. Then m is 2-magic. Consequently, each sequence of permutations gives rise to $(n!)^{d-1}$ matrices; and the number of pairs consisting of a matrix and a corresponding sequence of permutations equals $(n!)^{d-1}\, v_n(d)$.

Comparing these two expressions, we obtain (2.2), as required. □

Remark. We note that $u_1(d) = v_1(d) = 1$ for all d. Hence, Formula (2.2) is false for $n = 1$.

[2]??We warn the reader that for $d = 2$ this does not reduce to the notion of decomposability of matrices in linear algebra since there rows and columns are reordered by the *same* permutation.

3. Computation of $u_n(2)$ and $w_n(2)$

The number $w_n(2)$ of 2-dimensional 2-magic matrices of size n has been addressed earlier by Anand, Dumir and Gupta in [3, Sec. 8.1]. They found the generating function formula

$$\sum_{n\geq 0} w_n(2)\frac{z^n}{(n!)^2} = (1-z)^{-1/2}e^{z/2}. \tag{3.1}$$

This gives the explicit formula

$$w_n(2) = \sum_{k=0}^{n} \binom{2k}{k} \frac{(n!)^2}{2^{n-k}(n-k)!}. \tag{3.2}$$

Singularity analysis (cf. [9, Ch. VI]) applied to (3.1) then yields the asymptotic formula

$$w_n(2) = (n!)^2\sqrt{\frac{e}{\pi n}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \text{as } n\to\infty. \tag{3.3}$$

The number $u_n(2)$ of *indecomposable* 2-dimensional 2-magic matrices of size n can also be computed explicitly. One way is to observe that, by Birkhoff's Theorem (cf. [5] or [2, Corollary 8.40]), a 2-magic matrix m is the sum of two permutation matrices, say p_1 and p_2. If m is indecomposable, then the pair $\{p_1, p_2\}$ is uniquely determined. Premultiplying by p_1^{-1}, we obtain a situation where p_1 is the identity; indecomposability forces p_2 to be the permutation matrix corresponding to a cyclic permutation. So there are $n!\,(n-1)!$ choices for (p_1, p_2), and half this many choices for m (assuming, as we may, that $n > 1$). Note that this formula gives half the correct number for $n = 1$. So we have

$$u_n(2) = \begin{cases} 1, & \text{if } n = 1,\\ \frac{1}{2}n!\,(n-1)!, & \text{if } n > 1. \end{cases} \tag{3.4}$$

Alternatively, we may observe that 2-dimensional 2-magic matrices may be seen as a 2-sort species in the sense of Joyal [10] (see also [6, Def. 4 on p. 102]), with the row indices and the column indices forming the two set on which the functor defining the species operates. Hence, by the exponential principle for 2-sort species (see [10, ??] or [6, ??]), we have

$$\sum_{n\geq 0} w_n(2)\frac{z^n}{(n!)^2} = \exp\left(\sum_{n\geq 1} u_n(2)\frac{z^n}{(n!)^2}\right). \tag{3.5}$$

Combining this with (3.1), we find that

$$\sum_{n\geq 1} u_n(2)\frac{z^n}{(n!)^2} = \frac{z}{2} + \frac{1}{2}\log\left(\frac{1}{1-z}\right).$$

Extraction of the coefficient of z^n then leads (again) to (3.4).

4. A recurrence relation for $v_n(d)$

In this section we prove a recurrence relation for the numbers $v_n(d)$ (see Section 2 for their definition). By Lemma 1, this affords as well a recurrence relation for the numbers $u_n(d)$.

Proposition 2. *The numbers $v_n(d)$ satisfy $v_1(d) = 1$ and*

$$\sum_{k=1}^{n} \binom{n-1}{k-1}\left((2n-2k-1)!!\right)^{d-1} v_k(d) = \left((2n-1)!!\right)^{d-1}, \quad n > 1. \tag{4.1}$$

Here, $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ *is the product of the first* n *odd positive integers for* $n > 0$*, and, by convention,* $(-1)!! = 1$*.*

Proof. Recall that $(2n-1)!!$ is the number of fixed-point-free involutions on a set of size $2n$. The number of choices of involutions $t_1, t_2, \ldots, t_d$, where t_1 is as in (2.1), such that the G-orbit containing 1 has size $2k$ is

$$\binom{n-1}{k-1} \left((2n-2k-1)!!\right)^{d-1} v_k(d),$$

since we can choose in order

(i) $k-1$ of the $n-1$ cycles of t_1 other than $(1,2)$ such that the elements not fixed by all of these $k-1$ transpositions together with $\{1,2\}$ form the desired orbit, O say;
(ii) $d-1$ fixed-point-free involutions on O which, together with the restriction of t_1 to O, generate a transitive group;
(iii) $d-1$ arbitrary fixed-point-free involutions on the complement of O.

Summing these values shows that the numbers $v_n(d)$ satisfy the desired recurrence. □

Corollary 3. *For all integers* $d > 1$*, the numbers* $u_n(d)$ *satisfy* $u_1(d) = 1$ *and*

$$\left((2n-3)!!\right)^{d-1} + \sum_{k=2}^{n} \binom{n-1}{k-1} \left(\frac{(2n-2k-1)!!}{k!}\right)^{d-1} 2^k\, u_k(d) = \left((2n-1)!!\right)^{d-1}, \qquad n > 1.$$

Remarks. (1) In the case $d = 2$, we have seen in (3.4) that $u_n(2) = n!(n-1)!/2$ for $n > 1$, so that

$$v_n(2) = 2^{n-1}\,(n-1)! = (2n-2)!!,$$

where $(2n-2)!!$ is the product of the even integers up to $2n-2$ (with $0!! = 1$ by convention). Substituting this in (4.1), we have proved the somewhat curious looking identity

$$\sum_{k=1}^{n} \binom{n-1}{k-1} (2n-2k-1)!!\,(2k-2)!! = (2n-1)!!$$

for $n > 1$. It is, however, just an instance of the Chu–Vandermonde identity, which becomes obvious when the left-hand side is written in standard hypergeometric notation (cf. e.g. [12, (1.7.7), Appendix (III.4)]) as

$$2^{n-1}\,(1/2)_{n-1} \cdot {}_2F_1\left[\begin{matrix} -n+1, 1 \\ -n+\frac{1}{2} \end{matrix}; 1\right].$$

(2) For $d > 2$, we have not been able to solve the recurrence explicitly. However, it is easy to calculate terms in the sequences, and we can describe their asymptotics (see Sections 5 and 6).

Table 1 gives counts of all indecomposable matrices, all zero-one matrices, and all non-negative integer matrices, with dimension d and hyperplane sums 2. The sequences for $d = 2$ are numbers A010796, A001499, and A000681 in the On-Line Encyclopedia of Integer Sequences [13]. For $d = 3$, they are A112578, A112579 and A112580.

d	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
2	1	1	6	72	1440	43200
	0	1	6	90	2040	67950
	1	3	21	282	6210	202410
3	1	8	900	359424	370828800	820150272000
	0	8	900	366336	378028800	833156928000
	1	12	1152	431424	427723200	920031955200

TABLE 1

5. Asymptotics of the numbers $u_n(d)$

This section provides the preparation for the determination of the asymptotics of the numbers $w_n(d)$ for $d \geq 3$ in the next section. Our goal here is to establish an asymptotic estimate for the sequence $u_n(d)$ with fixed $d \geq 3$.

Proposition 4. *For fixed $d \geq 3$, we have*

$$u_n(d) \sim 2^{-dn}((2n)!)^{d-1}, \qquad \text{as } n \to \infty.$$

Proof. By Lemma 1, we have $u_n(d) = (n!)^{d-1} v_n(d)/2^n$ for $n > 1$, so it suffices to show that

$$v_n(d) \sim ((2n-1)!!)^{d-1}.$$

We will use the estimates

$$\sqrt{2(n+1)} \leq \frac{2^n\, n!}{(2n-1)!!} \leq 2\sqrt{n}$$

for $n \geq 1$. With $c_n = 2^n n!/(2n-1)!!$, we have $c_{n+1}/c_n = (2n+2)/(2n+1)$, and both inequalities are easily proved by induction. From these estimates, we obtain the inequality

$$\frac{(2n-1)!!}{(2k-1)!!\,(2n-2k-1)!!} \geq \binom{n}{k} \left(\frac{(k+1)(n-k+1)}{n}\right)^{1/2}. \tag{5.1}$$

To simplify our formulae, we denote the left-hand side of this inequality by $\left(\!\!\binom{n}{k}\!\!\right)$.

Now, by Proposition 4.1, $v_n(d)$ satisfies the recurrence

$$v_n(d) = ((2n-1)!!)^{d-1} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} ((2n-2k-1)!!)^{d-1} v_k(d), \qquad n > 1.$$

Clearly $v_n(d) \leq ((2n-1)!!)^{d-1}$. We show that $v_n(d) \geq ((2n-1)!!)^{d-1}(1 - O(1/n))$, an estimate which, in view of the above recurrence, would follow from

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(\!\!\binom{n}{k}\!\!\right)^{-(d-1)} = O\left(\frac{1}{n}\right).$$

Using (5.1), the quantity L on the left satisfies

$$L \le \frac{n}{(2n-1)^{d-1}} + \sum_{k=2}^{n-2} \binom{n-1}{k-1} \binom{n}{k}^{-(d-1)} \left(\frac{n}{(k+1)(n-k+1)} \right)^{(d-1)/2}$$
$$\le \frac{n}{(2n-1)^{d-1}} + \sum_{k=2}^{n-2} \frac{k}{n} \binom{n}{k}^{-(d-2)} \left(\frac{n}{(k+1)(n-k+1)} \right)^{(d-1)/2}.$$

Since $k/n < 1$, $n/(k+1)(n-k+1) < 1/2$, and $\binom{n}{k} \ge \binom{n}{2}$, and there are fewer than $n-1$ terms in the sum, the second term is at most

$$n^{-(d-2)}(n-1)^{-(d-3)} \cdot 2^{d-2} \cdot 2^{-(d-1)/2} \le \frac{1}{n},$$

as required. □

6. Asymptotics of the numbers $w_n(d)$

Recall that $w_n(d)$ and $u_n(d)$ are the numbers of unrestricted, respectively indecomposable, d-dimensional 2-magic matrices of size n. Using the exponential principle, we can relate the sequence $(w_n(d))_{n\ge 0}$ to the sequence $(u_n(d))_{n\ge 0}$ for each fixed d, see (6.1) below. This relationship combined with the fact that the sequence $(u_n(d))_{n\ge 0}$ grows sufficiently rapidly for $d \ge 3$ (Proposition 4 says that it grows very roughly like $((2n)!)^{d-1}$) allows us to conlude that, for $d \ge 3$, $w_n(d)$ and $u_n(d)$ grow at the same rate.

Theorem 5. *For fixed $d \ge 3$, we have*

$$w_n(d) \sim 2^{-nd}((2n)!)^{d-1}, \qquad \text{as } n \to \infty.$$

Proof. Generalizing the argument at the end of Section 3, we observe that d-dimensional 2-magic matrices may be seen as a d-sort species in the sense of Joyal [10] (see also [6, Def. 4 on p. 102]), with the row indices and the column indices forming the two set on which the functor defining the species operates. Hence, by the exponential principle for d-sort species (see [10, ??] or [6, ??]), we have

$$\sum_{n\ge 0} w_n(d) \frac{z^n}{(n!)^d} = \exp\left(\sum_{n\ge 1} u_n(d) \frac{z^n}{(n!)^d} \right).$$

If we now differentiate both sides of this equation with respect to z and subsequently multiply both sides by z, then we obtain

$$\sum_{n\ge 0} n w_n(d) \frac{z^n}{(n!)^d} = \left(\sum_{n\ge 1} n u_n(d) \frac{z^n}{(n!)^d} \right) \left(\sum_{n\ge 0} w_n(d) \frac{z^n}{(n!)^d} \right).$$

Comparison of coefficients of z^n on both sides then leads to the relation

$$w_n(d) = u_n(d) + \sum_{k=1}^{n-1} \frac{k}{n} \binom{n}{k}^d u_k(d) w_{n-k}(d). \tag{6.1}$$

As we said at the beginning of this section, our goal is to show that $w_n(d)$ grows asymptotically at the same rate as $u_n(d)$. Hence, putting $w_n(d) = u_n(d) + x_n(d)$, we have to show that $x_n(d) = o(u_n(d))$. We assume inductively that

$$x_m(d) \le 2^{-m}((2m-1)!!)^{d-1}(m!)^{d-1}$$

for all m between 2 and $n-1$; the induction starts since we have $x_1(d) = x_2(d) = 0$.

Now, using the inductive hypothesis with the recurrence relation (6.1), we have

$$\begin{aligned}\frac{x_n(d)2^n}{((2n-1)!!)^{d-1}(n!)^{d-1}} &\le 2\sum_{k=1}^{n-1}\frac{k}{n}\binom{n}{k}^{d}\left(\!\!\binom{n}{k}\!\!\right)^{-(d-1)}\binom{n}{k}^{-(d-1)}\\ &\le 2\sum_{k=1}^{n-1}\frac{k}{n}\binom{n}{k}^{-(d-2)}\left(\frac{n}{(k+1)(n-k+1)}\right)^{(d-1)/2}\\ &\le (2^{1/2}n)^{-(d-3)},\end{aligned}$$

which establishes the result if $d>3$. For $d=3$, this inequality gives the inductive step (that is, that the left-hand side is at most 1); the fact that it is $o(1)$ for large n is proved by an argument like that in the proof of Proposition 4. □

References

[1] M. Ahmed, J. De Loera and R. Hemmecke, Polyhedral cones of magic cubes and squares, in: Discrete and Computational Geometry. The Goodman–Pollack Festschrift (B. Aronov, S. Basu, J. Pach and M. Sharir, eds.), Springer–Verlag, Berlin, Algorithms Comb. 25, 2003, pp. 25–41.

[2] M. Aigner, *Combinatorial Theory,* Springer–Verlag, Berlin, 1979.

[3] H. Anand, V. C. Dumir, and H. Gupta, A combinatorial distribution problem, *Duke Math. J.* **33** (1966), 757–769.

[4] M. Beck, M. Cohen, J. Cuomo and P. Gribelyuk, The number of "magic" squares, cubes, and hypercubes, *Amer. Math. Monthly* **110** (2003), 707–717.

[5] G. Birkhoff, Tres observaciones sobre el algebra lineal, *Univ. Nac. Tucumán Rev. Ser. A* **5** (1946), 147–150.

[6] F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-Like Structures*, Cambridge University Press, Cambridge, 1998.

[7] P. J. Cameron, C. Krattenthaler and T. W. Müller, Decomposable functors and the exponential principle, II, *Séminaire Lotharingien Combin.* (to appear).

[8] J. A. De Loera, F. Liu and R. Yoshida, A generating function for all semi-magic squares and the volume of the Birkhoff polytope, *J. Algebraic Combin.* **30** (2009), 113–139.

[9] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[10] A. Joyal, Une théorie combinatoire des séries formelles, *Adv. in Math.* **42** (1981), 1–82.

[11] P. A. MacMahon, *Combinatory Analysis*, vol. 2, Cambridge University Press, 1916; reprinted by Chelsea, New York, 1960.

[12] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[13] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, available at `http://www.research.att.com:80/~njas/sequences/`.

[14] R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, California, 1986; reprinted by Cambridge University Press, Cambridge, 1998.

School of Mathematical Sciences, Queen Mary & Westfield College, University of London, Mile End Road, London E1 4NS, United Kingdom.
WWW: `http://www.maths.qmw.ac.uk/~pjc/`, `http://www.maths.qmw.ac.uk/~twm/`.

Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Vienna, Austria. WWW: `http://www.mat.univie.ac.at/~kratt`.