# Matrix groups

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## 1 Matrix groups and group representations

These two topics are closely related. Here we consider some particular groups which arise most naturally as matrix groups or quotients of them, and special properties of matrix groups which are not shared by arbitrary groups. In representation theory, we consider what we learn about a group by considering all its homomorphisms to matrix groups.

This article falls roughly into two parts. In the first part we discuss properties of specific matrix groups, especially the general linear group (consisting of all invertible matrices of given size over a given field) and the related "classical groups". In the second part, we consider what we learn about a group if we know that it is a linear group.

Most group theoretic terminology is standard and can be found in any textbook. A few terms we need are summarised in the next definition.

Let $X$ and $Y$ be group-theoretic properties. We say that a group $G$ is **locally X** if every finite subset of $G$ is contained in a subgroup with property X; $G$ is **X-by-Y** if $G$ has a normal subgroup $N$ such that $N$ has X and $G/N$ has Y; and $G$ is **poly-X** if $G$ has subgroups $N_0 = 1, N_1, \ldots, N_r = G$ such that, for $i = 0, \ldots, r-1$, $N_i$ is normal in $N_{i+1}$ and the quotient has $X$. Thus a group is locally finite if and only if every finitely generated subgroup is finite; and a group is solvable if and only if it is poly-abelian.

## 2 Introduction

This section contains some basic definitions about matrix groups.

A **matrix group**, or **linear group**, is a group $G$ whose elements are invertible $n \times n$ matrices over a field $F$. The **general linear group** $\mathrm{GL}(n,F)$ is the group consisting of all invertible $n \times n$ matrices over the field $F$. So a group $G$ is a matrix group precisely when it is a subgroup of $\mathrm{GL}(n,F)$ for some natural number $n$ and field $F$. If $V$ is a vector space of dimension $n$ over $F$, we denote by $\mathrm{GL}(V)$ the group of invertible linear transformations of $V$; thus $\mathrm{GL}(V) \cong \mathrm{GL}(n,F)$.

Which groups are isomorphic to matrix groups? Auslander and Swan showed that every polycyclic group is linear over $\mathbb{Z}$; conversely, a solvable linear group over $\mathbb{Z}$ is polycyclic.

Mal'cev showed that linearity of bounded degree is a local property:

**Theorem 2.1** *If every finitely generated subgroup of a group G is isomorphic to a linear group of degree n, then G is isomorphic to a linear group of degree n. The same assertion holds in any given characteristic.*

**Fact 2.2** *Any free group is linear of degree* 2 *in every characteristic. More generally, a free product of linear groups is linear.*

A matrix group $G \leq \mathrm{GL}(V)$ is said to be **reducible** if there is a $G$-invariant subspace $U$ of $V$ other than $\{0\}$ and $V$, and is **irreducible** otherwise. A matrix group $G \leq \mathrm{GL}(V)$ is said to be **decomposable** if $V$ is the direct sum of two non-zero $G$-invariant subspaces, and is **indecomposable** otherwise. If $V$ can be expressed as the direct sum of irreducible subspaces, then $G$ is **completely reducible**. (Thus an irreducible group is completely reducible.) If the matrix group $G \leq \mathrm{GL}(n,F)$ is irreducible regarded as a subgroup of $\mathrm{GL}(n,K)$ for any algebraic extension $K$ of $F$, we say that $G$ is **absolutely irreducible**.

**Theorem 2.3 (Maschke's Theorem)** *Let G be a finite linear group over F, and suppose that the characteristic of F is either zero or coprime to $|G|$. If G is reducible, then it is decomposable.*

The image of a representation of a group is a matrix group. We apply descriptions of the matrix group to the representation: thus, if $\rho : G \to \mathrm{GL}(V)$ is a representation, and $G\rho$ is irreducible, indecomposable, absolutely irreducible, etc., then we say that the representation $\rho$ is irreducible, etc.

Maschke's Theorem immediately extends to show that a locally finite group in characteristic zero is completely reducible.

**Theorem 2.4 (Clifford's Theorem)** *Let G be an irreducible linear group on a vector space V of dimension n, and let N be a normal subgroup of G. Then V is a direct sum of minimal N-spaces $W_1, \ldots, W_d$ permuted transitively by G. In particular, d divides n; the group N is completely reducible; and the linear groups induced on $W_i$ by N are all isomorphic.*

It follows that a normal (or even a subnormal) subgroup of a completely reducible linear group is completely reducible.

In general, a linear group $G$ on $V$ has a **unipotent** normal subgroup $U$ such that $G/U$ is isomorphic to a completely reducible linear group on $V$; the subgroup

$U$ is nilpotent of class at most $n-1$, where $n = \dim(V)$. (A matrix is unipotent if all its eigenvalues are 1; a linear group is unipotent if all its elements are. A unipotent linear group is conjugate (in the general linear group) to a group of upper unitriangular matrices.)

A theorem of Mal'cev asserts that complete reducibility is a local property:

**Theorem 2.5** *If every finitely generated subgroup of the linear group $G$ is completely reducible, then $G$ is completely reducible.*

Over an algebraically closed field, we have the following algebraic criterion for irreducibility:

**Theorem 2.6** *Let $G$ be a linear group on $F^n$, where $F$ is algebraically closed. Then $G$ is irreducible if and only if the elements of $G$ span the space of all $n \times n$ matrices over $F$.*

# 3 The general and special linear groups

In this section we consider the largest matrix group for given size of matrices and given field, the general linear group, and its normal subgroup the special linear group.

If $F$ is the finite field of order $q$ (the Galois field $\mathrm{GF}(q)$), then we write $\mathrm{GL}(n,q)$ for $\mathrm{GL}(n,F)$.

The **special linear group** $\mathrm{SL}(n,F)$ is the normal subgroup of $\mathrm{GL}(n,F)$ consisting of matrices of determinant 1. The **projective general linear group** and **projective special linear group** $\mathrm{PGL}(n,F)$ and $\mathrm{PSL}(n,F)$ are the quotients of $\mathrm{GL}(n,F)$ and $\mathrm{SL}(n,F)$ by their normal subgroups $Z$ and $Z \cap \mathrm{SL}(n,F)$, respectively, where $Z$ is the group of non-zero scalar matrices. If $F = \mathrm{GF}(q)$, we write $\mathrm{SL}(n,q)$ for $\mathrm{SL}(n,F)$, etc.

**Theorem 3.1** *Given any two ordered bases for the vector space $V$, there is a unique element of $\mathrm{GL}(V)$ carrying the first to the second.*

From this theorem it follows that the order of $\mathrm{GL}(n,q)$ is equal to the number of ordered bases of $\mathrm{GF}(q)^n$, namely

$$|\mathrm{GL}(n,q)| = \prod_{i=0}^{n-1}(q^n - q^i) = q^{n(n-1)/2}\prod_{i=0}^{n-1}(q^{n-i} - 1).$$

A **transvection** is a linear transformation $t$ on $V$ with all eigenvalues equal to 1 and satisfying $\mathrm{rank}(t-1) = 1$. A transvection has the form

$$t : x \mapsto x + (xf)v$$

where $v \in V$, $f \in V^*$, and $vf = 0$. Note that transvections have determinant 1, and so lie in $\mathrm{SL}(V)$.

**Theorem 3.2** *The group* $\mathrm{SL}(n,F)$ *is generated by transvections, for any* $n \geq 2$ *and any field* $F$*.*

**Theorem 3.3** *The group* $\mathrm{PSL}(n,F)$ *is simple for all* $n \geq 2$ *and all fields* $F$*, except for the two cases* $\mathrm{PSL}(2,2)$ *and* $\mathrm{PSL}(2,3)$.

**Fact 3.4** *Basic facts from linear algebra about similarity of matrices can be interpreted as statements about conjugacy classes in* $\mathrm{GL}(n,F)$*. For example, two non-singular matrices are conjugate in* $\mathrm{GL}(n,F)$ *if and only if they have the same invariant factors. If* $F$ *is algebraically closed, then two non-singular matrices are conjugate in* $\mathrm{GL}(n,F)$ *if and only if they have the same Jordan form.*

# 4 The BN structure of the general linear group

The general linear groups $\mathrm{GL}(n,F)$, for $n \geq 2$, contain a configuration of subgroups known as a **BN-pair** or **Tits system**. Our treatment is descriptive, but an axiomatic approach is also possible.

Let $G$ be the general linear group $\mathrm{GL}(n,F)$. Let $B$ be the group of upper-triangular matrices in $G$; $U$ the group of strictly upper-triangular matrices (with diagonal entries 1); $H$ the group of diagonal matrices; and $N$ the normaliser of $H$ in $G$, the group of monomial matrices (those having a unique non-zero element in each row or column). The following properties are easily verified:

- $B = UH$, $B \cap N = H$;
- $N/H$ is isomorphic to the symmetric group $S_n$.

Let $s_i$ be the transposition $(i, i+1)$ in $S_n$, for $i = 1, \ldots, n$. We can take as coset representative of $s_i$ in $N$ the reflection which interchanges the $i$th and $(i+1)$st basis vectors; we denote this element also by $s_i$. The elements $s_1, \ldots, s_{n-1}$ generate $S_n$, and indeed $S_n$ is a **Coxeter group**: a presentation is given by the relations $s_i^2 = 1$ (for $i = 1, \ldots, n-1$), $(s_i s_{i+1})^3 = 1$ (for $i = 1, \ldots, n-2$); and $(s_i s_j)^2 = 1$ if $|j - i| > 1$.

Now it can be shown that $G = BNB$. Moreover, there are $n!$ double cosets $BxB$; we can take as double coset representatives the elements of $S_n$ (regarded as products of the generators $s_i$). Furthermore, for any $w \in S_n$ and $i = 1, \ldots, n-1$, we have

$$BwBs_iB \subseteq BwB \cup Bws_iB,$$

from which it follows that $\langle B, s_1, \ldots, s_{n-1} \rangle = G$.

The subgroups $B$ and $W = N/H$ are known as the **Borel subgroup** and **Weyl group** of $G$. From the relation above, many strong properties can be deduced: here is an example.

**Theorem 4.1** *Any subgroup of G containing B has the form*

$$P_I = \langle B, s_i : i \in I \rangle$$

*for some subset I of* $\{1, \ldots, n-1\}$. *Thus there are* $2^{n-1}$ *such subgroups.*

The subgroups $P_I$ are the **parabolic subgroups** of $G$ (relative to the given BN-pair). The maximal parabolic subgroups are those for which $I = \{1, \ldots, n-1\} \setminus \{j\}$ for some $j$; it is easy to see that in this case $P_I$ is the stabiliser of the subspace spanned by the last $n - j$ basis vectors.

More generally, with respect to any basis of $V$ there is a BN-structure. We use the terms "Borel subgroup" and "parabolic subgroup" to refer to the subgroups defined with respect to an arbitrary basis. By Theorem 3.1, for example, all the Borel subgroups of $\mathrm{GL}(V)$ are conjugate. The maximal parabolic subgroups are precisely the maximal reducible subgroups.

# 5 Classical groups

The classical groups form several important families of linear groups. We give a brief description here, and refer to the books [7, 17] or the article [9] for more details.

Let $\sigma$ be an automorphism of $F$. A **$\sigma$-sesquilinear form** on the $F$-vector space $V$ is a function $B : V \times V \to F$ which is linear as a function in the first variable and $\sigma$-semilinear in the second. If $\sigma$ is the identity, then $B$ is a **bilinear form**. A sesquilinear form $B$ is **non-degenerate** if $B(v, w) = 0$ for all $w \in V$ implies $v = 0$. A sesquilinear form $B$ is **$\sigma$-Hermitian** if $B(v, w) = B(w, v)^\sigma$ for all $v, w \in V$. A bilinar form $B$ is **alternating** if $B(v, v) = 0$ for all $v \in V$.

In general, the **radical** of $B$ is the subspace $V^\perp = \{v \in V : B(v, w) = 0$ for all $w \in V$. In general one can define a left and right radical, but for alternating or Hermitian forms the two radicals coincide.

A **quadratic form** on the $F$-vector space $V$ is a function $Q : V \to F$ satisfying

- $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in F$, $v \in V$;
- the function $B : V \times V \to F$ defined by

  $$B(v, w) = Q(v + w) - Q(v) = Q(w)$$

  is bilinear.

The quadratic form $Q$ is **non-singular** if $Q(v) = 0$ and $B(v,w) = 0$ for all $w \in V$ imply that $v = 0$; equivalently, the form $Q$ is non-zero on all vectors in the radical of $B$ except the zero vector.

**Fact 5.1** *If the characteristic of $F$ is not $2$, then the form $B$ is symmetric (that is, $1$-Hermitian), and $Q$ can be recovered from $B$ by the formula $Q(v) = \frac{1}{2}B(v,v)$. If the characteristic is $2$, however, the form $B$ is alternating and does not determine $Q$; there are many quadratic forms associated with a given alternating form.*

A **classical group** over $V$ is the subgroup of $\mathrm{GL}(V)$ consisting of transformations preserving either a non-degenerate Hermitian or alternating form or a non-singular quadratic form. We distinguish three types of classical groups:

**orthogonal group:** preserving a non-singular quadratic form $Q$;

**symplectic group:** preserving a non-degenerate alternating bilinear form $B$;

**unitary group:** preserving a non-degenerate $\sigma$-Hermitian form $B$, with $\sigma \neq 1$.

We denote a classical subgroup of $\mathrm{GL}(V)$ by $\mathrm{O}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{U}(V)$ depending on type. If necessary, we add extra notation to specify which particular form is being used. If $V = F^n$, we also write $\mathrm{O}(n,F)$, $\mathrm{Sp}(n,F)$ or $\mathrm{U}(n,F)$.

A **formed space** will denote a vector space carrying a form of one of the three classical types specified above.

The **Witt index** of a formed space $V$ is the dimension of the largest subspace on which the form is identically zero. We also speak of the Witt index of the corresponding classical group.

Classification of classical groups up to conjugacy in $\mathrm{GL}(n,F)$ is equivalent to classification of forms of the appropriate type up to the natural action of the general linear group together with scalar multiplication. Often this is a very difficult problem: the next theorem gives a few cases where the classification is more straightforward.

**Theorem 5.2** *(a) A non-degenerate alternating form on $V = F^n$ exists if and only if $n$ is even, and all such forms are equivalent. So there is a unique conjugacy class of symplectic groups in $\mathrm{GL}(n,F)$ if $n$ is even (with Witt index $n/2$), and none if $n$ is odd.*

*(b) Let $F = \mathrm{GF}(q)$. Then, up to conjugacy, $\mathrm{GL}(n,q)$ contains one conjugacy class of unitary subgroups (with Witt index $\lfloor n/2 \rfloor$), one class of orthogonal subgroups if $n$ is odd (with Witt index $(n-1)/2$), and two classes if $n$ is even (with Witt indices $n/2$ and $n/2-1$).*

*(c) A non-singular quadratic form on* $\mathbb{R}^n$ *is determined up to the action of* $\mathrm{GL}(n,\mathbb{R})$ *by its signature. Its Witt index is* $\min\{s,t\}$*, where* $s$ *and* $t$ *are the numbers of positive and negative eigenvalues. So there are* $\lfloor n/2 \rfloor + 1$ *conjugacy classes of orthogonal subgroups of* $\mathrm{GL}(n,\mathbb{R})$*, with Witt indices* $0,1,\ldots,\lfloor n/2 \rfloor$.

The analogue for the classical groups of Theorem 3.1 is Witt's Lemma. An **isometry** between subspaces of a formed space is a linear transformation preserving the value of the form.

**Theorem 5.3 (Witt's Lemma)** *Suppose that* $U_1$ *and* $U_2$ *are subspaces of the formed space* $V$*, and* $h : U_1 \to U_2$ *is an isometry. Then there is an isometry* $g$ *of* $V$ *which extends* $h$ *if and only if* $(U_1 \cap V^\perp)h = U_2 \cap V^\perp$.

*In particular, if* $V^\perp = 0$*, then any isometry between subspaces of* $V$ *extends to an isometry of* $V$*.*

From Witt's Lemma it is possible to write down formulae for the orders of the classical groups over finite fields similar to the formula we gave for the general linear group.

The analogues of Theorems 3.2 and 3.3 hold for the classical groups with non-zero Witt index. However, the situation is more complicated. Any symplectic transformation has determinant 1, so $\mathrm{Sp}(2r,F) \le \mathrm{SL}(2r,F)$. Moreover, $\mathrm{Sp}(2r,F)$ is generated by symplectic transvections (those preserving the alternating form) for $r \geq 2$, except for $\mathrm{Sp}(4,2)$. Similarly, the **special unitary group** $\mathrm{SU}(n,F)$ (the intersection of $\mathrm{U}(n,F)$ with $\mathrm{SL}(n,F)$) with positive Witt index is generated by unitary transvections (those preserving the Hermitian form), except for $\mathrm{SU}(3,2)$.

There are no "orthogonal transvections" except in characteristic 2, and it is necessary to use the more complicated **long root elements** instead.

Another complication with the orthogonal groups is that the group generated by these elements is not always the intersection of the orthogonal group with the special linear group, but may be smaller. In general it is denoted by $\Omega(n,F)$, with possibly some additional notation to specify which quadratic form is being considered.

Then it can be shown that, if the Witt index is at least 3, then the quotient of $\mathrm{Sp}(n,F)$, $\mathrm{SU}(n,F)$ or $\Omega(n,F)$ by the group of scalar matrices it contains is simple.

Like the general linear groups, the classical groups contain BN-pairs (configurations of subgroups satisfying conditions like those in the previous section). The difference is that the Weyl group $W = N/H$ is not the symmetric group, but one of the other types of Coxeter group (finite groups generated by reflections).

Although our treatment of classical groups has been as far as possible independent of fields, it is worth making two remarks here.

- The theory can be extended to classical groups over rings. This has important connections with algebraic K-theory. The book [7] gives details.

- On the other hand, for most of mathematics, the classical groups over the real and complex numbers are the most important, and among these, the real orthogonal and complex unitary groups preserving positive definite forms take pride of place: see [24].

A representation of a group $G$ (over the complex numbers) is said to be **unitary** if its image is contained in the unitary group (that is, preserves a positive definite Hermitian form). It is known that every representation of a finite group is equivalent to a unitary representation.

# 6 Aschbacher's Theorem

Aschbacher's Theorem gives a description of the maximal subgroups of the classical groups over finite fields. We state the result here just for $\mathrm{GL}(n,q)$, and refer to [1, 10] for the general case. The theorem is the analogue for linear groups of the O'Nan–Scott Theorem for permutation groups.

We define eight classes of subgroups of $\mathrm{GL}(n,q)$ as follows:

$\mathscr{C}_1$: Stabilisers of subspaces of $V$. (These are precisely the maximal parabolic subgroups of $\mathrm{GL}(n,q)$.)

$\mathscr{C}_2$: Stabilisers of direct sum decompositions of $V$ into subspaces of the same dimension.

$\mathscr{C}_3$: Semilinear groups over extension fields of $\mathrm{GF}(q)$ of prime degree: that is, the product of $\mathrm{GL}(m,q^k)$ and the Galois group of $\mathrm{GF}(q^k)$ over $\mathrm{GF}(q)$ inside $\mathrm{GL}(mk,q)$, where $k$ is prime.

$\mathscr{C}_4$: Stabilisers of tensor product decompositions $V = V_1 \otimes V_2$.

$\mathscr{C}_5$: Linear groups over subfields of $\mathrm{GF}(q)$ of prime index.

$\mathscr{C}_6$: Normalisers of $r$-groups of symplectic type, where $r$ is a prime different from $p$.

$\mathscr{C}_7$: Stabilisers of tensor product decompositions $V = \bigotimes_{i=1}^{t} V_i$, where the $V_i$ all have the same dimension.

$\mathscr{C}_8$: Classical groups.

Here, for a prime number $r$, an $r$-group is said to have **symplectic type** if it is a central product of a cyclic group and an extraspecial group. The normaliser of this group involves a symplectic group (if $r$ is odd) or an orthogonal group (if $r$ is even); we assume that the action of this group is irreducible. The definition of the other classes should be obvious.

**Theorem 6.1 (Aschbacher's Theorem for** $\mathrm{GL}(n,q)$**)** *Let $H$ be a subgroup of $\mathrm{GL}(n,q)$, not containing $\mathrm{SL}(n,q)$. Then either*

*(a) $H$ is contained in a member of one of the classes $\mathscr{C}_1$–$\mathscr{C}_8$; or*

*(b) $H$ is absolutely irreducible and $H$ modulo scalars is almost simple.*

Liebeck [12] proved the following addition, showing that the groups in case (b) are very small (recall that the order of $\mathrm{GL}(n,q)$ is roughly $q^{n^2}$).

**Theorem 6.2** *In case (b) of Aschbacher's Theorem, $|G| \le q^{3n}$.*

# 7 Finite matrix groups

According to Cayley's Theorem, every group is isomorphic to a permutation group (a subgroup of a symmetric group). However, not every group is isomorphic to a matrix group (a subgroup of a general linear group). In the remainder of the survey, we will give some results about the structure of matrix groups. The information will be divided into several sections: properties of finite and finitely generated matrix groups; periodic matrix groups; and solvable and nilpotent matrix groups. We conclude with a brief look at finitary groups, a generalization of matrix groups.

References on matrix groups (most of them fairly old) include Dixon [6], Suprunenko [16], and Wehrfritz [22].

We begin by observing that every finite group is isomorphic to a matrix group (we can embed it in a finite symmetric group and then take the representation by permutation matrices). However, the class of matrix groups of fixed degree in characteristic zero is restricted by a theorem of Jordan:

**Theorem 7.1** *There is a function $f$ on the natural numbers such that, if $G$ is a finite linear group of degree $n$ over a field of characteristic zero (or not dividing $|G|$), then $G$ has an abelian normal subgroup of index at most $f(n)$.*

The function $f(n)$ can be taken to be $(cn)^{n^2}$ for some constant $c$.

The precise analogue for non-zero characteristic is false: the group $\mathrm{GL}(n,q)$ has order roughly $q^{n^2}$ and any abelian normal subgroup is contained in the centre.

However, Brauer and Feit [4] showed that there is an abelian normal subgroup whose index is bounded by a function of $n$ and the order of a Sylow $p$-subgroup of $G$, where $p$ is the characteristic.

Burnside proved that any finite subgroup of $\mathrm{GL}(n,\mathbb{Q})$ is conjugate to a subgroup of $\mathrm{GL}(n,\mathbb{Z})$. More generally, a finite subgroup of $\mathrm{GL}(n,F)$ is conjugate to a subgroup of $\mathrm{GL}(n,R)$ if $R$ is a principal ideal domain with field of fractions $F$.

# 8 Finitely generated matrix groups

Perhaps the most important general result on finitely generated matrix groups is the **Tits alternative** (Tits [20]). This shows that in a certain sense a finitely generated matrix group is either "tame" or "wild".

**Theorem 8.1** *A finitely generated linear group either is solvable-by-finite, or contains a nonabelian free subgroup.*

This theorem has been the starting point for similar results for a number of other classes of groups, including linear groups over division rings, mapping class groups, groups associated with low-dimensional manifolds, and groups that are residually of bounded rank. An algorithmic form of Tits' theorem has been given by Beals [2].

Among the many consequences of the Tits alternative, we mention just two. The first is the observation of Tits himself that the "growth" of a finitely generated linear group (the growth rate for the number of group elements which are words of length $n$ in the generators) is either polynomial or exponential. (For general groups, an example of Grigorchuk shows that growth strictly between polynomial and exponential is possible.) The second is a striking theorem of Larsen and Lubotzky [11]:

**Theorem 8.2** *Let $G$ be a finitely generated residually finite linear group over an algebraically closed field. Suppose that the number of normal subgroups of $G$ of index at most $n$ is bounded by a polynomial in $n$. Then either $G$ is solvable-by-finite, or $G$ involves a product of simple algebraic groups of type $G_2$, $F_4$ or $E_8$.*

We conclude with a couple more special properties of finitely generated matrix groups.

**Theorem 8.3** *Let $G$ be a finitely generated linear group.*

- *(Mal'cev) $G$ is Hopfian, that is, not isomorphic to any proper quotient;*
- *(Rabin) $G$ has solvable word problem.*

# 9 Periodic matrix groups

Burnside showed that a finitely generated linear group of finite exponent in characteristic zero is finite. The result extends to arbitrary characteristic as follows:

**Theorem 9.1** *Let $G$ be a linear group of degree $n$ over $F$, and suppose that $G$ has exponent $e$. Then $G$ has a normal subgroup $N$ of index at most $e^{n^3}$ such that*

- *if* $\mathrm{char}(F) = 0$, *then* $N = 1$;
- *if* $\mathrm{char}(F) = p > 0$, *then $N$ is a $p$-group of nilpotency class at most $n-1$ consisting of unipotent elements. In particular, $N = 1$ if $G$ is completely reducible.*

This shows that a completely reducible linear group of finite exponent is locally finite. But more is true, as was shown by Schur in characteristic zero and Kaplansky in general:

**Theorem 9.2** *A periodic linear group is locally finite.*

Using this result, Jordan's theorem extends immediately from finite to periodic linear groups in characteristic zero (or not dividing the characteristic).

Winter [25] showed that a completely reducible periodic linear group over an algebraically closed field is conjugate to a linear group over the algebraic closure of the prime subfield. It follows immediately that any completely reducible periodic linear group is countable.

Wehrfritz [21] showed that analogues of Sylow's theorems hold for periodic linear groups: that is, for any prime $p$, such a group has maximal $p$-subgroups, and any two such subgroups are conjugate. Other results of P. Hall and Wielandt on Sylow theory for finite groups have been extended to this class of groups.

The simple locally finite matrix groups were determined by Thomas [18, 19] and Borovik [3]. The proof uses the Classification of Finite Simple Groups. The "groups of Lie type" include the classical linear groups and some exceptional families associated with the exceptional simple Lie algebras over $\mathbb{C}$ or with exceptional automorphisms of their Dynkin diagrams: see Carter [5] for discussion.

**Theorem 9.3** *A locally finite simple linear group is a group of Lie type.*

## 10 Solvable and nilpotent matrix groups

Mal'cev showed that the analogue of Jordan's theorem holds for completely reducible solvable groups: such a group has a normal abelian subgroup whose index is bounded by a function of the degree. If we delete the assumption of complete reducibility, we can assert that there is a triangularisable subgroup of bounded index, so that the group is nilpotent-by-abelian-by-finite.

Zassenhaus proved that a solvable linear group of degree $n$ has derived length bounded by a function $\rho$ of $n$. (The best possible bounds for $\rho$, namely

$$5\log_9(n-1)+D \le \rho(n) \le 5\log_9(n-2)+D+\tfrac{3}{2},$$

for $n \ge 66$, where $D = \frac{17}{2} - (15\log 2)/(2\log 3)$, are due to Newman [14].) It follows that a locally solvable linear group is solvable.

The next theorem summarises some results of Platonov and Wehrfritz.

**Theorem 10.1** *Let G be a finitely generated linear group. Then*

- *a locally nilpotent subgroup of G is nilpotent;*
- *the Frattini subgroup of G is nilpotent;*
- *the Fitting subgroup of G is the set of all right Engel elements of G.*

Suprunenko showed that the centre of an irreducible nilpotent linear group has index bounded by a function of the degree and the nilpotency class.

## 11 Finitary groups

I conclude with a brief account of a generalization of matrix groups which was first introduced by Dieudonné but has had a lot of attention recently. Let $V$ be an infinite-dimensional vector space over a field $F$. An invertible linear transformation $g$ on $V$ is called **finitary** if the image of $g-1$ has finite dimension (equivalently, the subspace of fixed points of $g$ has finite codimension). A group of linear transformations of $V$ is called **finitary** if all its elements are finitary.

A highlight of the theory of finitary groups is a result of Hall [8], which extends Thomas' result (Theorem 9.3) on linear groups.

**Theorem 11.1** *A simple locally finite finitary group is either an alternating group or a finitary analogue of a "classical" matrix group.*

Wehrfritz has generalized many basic results on complete reducibility, Clifford theory, etc., of linear groups to finitary groups. He has also extended the concept of finitary groups from vector spaces to modules over rings. A rich and complex theory emerges: see [23] for some results.

A survey of finitary groups appears in Phillips [15].

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