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ORBIT-HOMOGENEITY

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Abstract

We introduce the concept of orbit-homogeneity of permutation groups: a group G is orbit t-homogeneous if two sets of cardinality t lie in the same orbit of G whenever their intersections with each G-orbit have the same cardinality. For transitive groups, this coincides with the usual notion of t-homogeneity. This concept is also compatible with the idea of partition transitivity introduced by Martin and Sagan.

We show that any group generated by orbit t-homogeneous subgroups is orbit t-homogeneous, and that the condition becomes stronger as t increases up to $\lfloor n/2 \rfloor$, where n is the degree. So any group G has a unique maximal orbit t-homogeneous subgroup $\Omega_t(G)$, and $\Omega_t(G) \leq \Omega_{t-1}(G)$.

We also give some structural results for orbit t-homogeneous groups and a number of examples.

A permutation group G acting on a set V is said to be *t-homogeneous* if it acts transitively on the set of t-element subsets of V. The t-homogeneous groups which are not t-transitive have been classified (see [**4**, **5**, **6**]); the classification of t-transitive groups for $t > 1$ follows from the classification of finite simple groups [**3**] (the list is given in [**1**]).

A permutation group G acting on a set V is said to be *orbit-t-homogeneous*, or *t-homogeneous with respect to its orbit decomposition*, if whenever S_1 and S_2 are t-subsets of V satisfying $|S_1 \cap V_i| = |S_2 \cap V_i|$ for every G-orbit V_i, there exists $g \in G$ with $S_1 g = S_2$. Thus, a group which is t-homogeneous in the usual sense is orbit-t-homogeneous; every group is orbit-1-homogeneous; and the trivial group is orbit-t-homogeneous for every t. It is also clear that a group of degree n is orbit-t-homogeneous if and only if it is orbit-$(n-t)$-homogeneous; so, in these cases, we may assume $t \leq n/2$ without loss of generality.

If two sets S_1 and S_2 are subsets of V satisfying $|S_1 \cap V_i| = |S_2 \cap V_i|$ for every G-orbit V_i then S_1 and S_2 are said to have the same structure with respect to G (or just to have the same structure if the group is obvious).

Theorem 4.3.4 of [**2**] is the following:

Theorem 1. *If G and H are orbit-t-homogeneous on V, then so is $\langle GH \rangle$.*

Young extended the concept of homogeneous groups by investigating the relationship between permutation groups and partitions [**8**]. A partition of V, $P = (P_1, P_2, \ldots, P_k)$, is said to have shape

$$|P| = (|P_1|, |P_2|, \ldots, |P_k|)\,.$$

A group element $g \in G$ is said to map the partition P onto a partition $Q = (Q_1, Q_2, \ldots, Q_k)$ if $P_i g = Q_i$ for all i. Obviously, a pre-requisite for this is that P and Q have the same structure with respect to G, i.e. that P_i and Q_i have the same

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structure for all i. The permutation group G is said to be orbit-λ-transitive if, for any two partitions of V that have shape λ and the same structure, P and Q say, there exists some $g \in G$ that maps P to Q. A permutation group of degree n is orbit-t-homogeneous if and only if it is orbit-λ-transitive, where $\lambda = (n-t, t)$.

The following is a more generalised version of Theorem 1.

THEOREM 2. *If G and H are orbit-λ-transitive on V, then so is $\langle GH \rangle$.*

Proof. Let P and Q be partitions of V that have the same structure with respect to $\langle GH \rangle$ and have shape λ. It is sufficient to show that there exists $\sigma \in \langle GH \rangle$ such that $P\sigma = Q$ when

- $P_1 = S_1 \cup \{x_1\}$ and $P_2 = S_2 \cup \{x_2\}$,
- $Q_1 = S_1 \cup \{x_2\}$ and $Q_2 = S_2 \cup \{x_1\}$,
- $P_i = Q_i$ for all $i > 2$,

for some $S_1, S_2 \subseteq V$. Since P and Q have the same structure with respect to $\langle GH \rangle$, x_1 and x_2 must lie in the same $\langle GH \rangle$-orbit and so there exists an element $\sigma' = g_1 h_1 \ldots g_m h_m$ such that $x_1 \sigma' = x_2$.

Suppose that $m = 1$ and let $y = x_1 g_1$. Note that x_1 and y lie in the same G-orbit and that y and x_2 lie in the same H-orbit. If $y = x_1$ or $y = x_2$ then result is obvious, so assume that is not the case. There are now several cases to deal with.

Suppose that $y \in P_1$, i.e. $S_1 = S_1' \cup \{y\}$, and consider the partition $R = (R_1, R_2, \ldots)$ where

$$R_1 = S_1' \cup \{x_1, x_2\}, \quad R_2 = S_2 \cup \{y\}, \quad R_i = P_i = Q_i \text{ for all } i > 2.$$

The partitions P and R have the same structure with respect to H and both have shape λ. Hence there exists $h \in H$ such that $Ph = R$. Similarly the partitions R and Q have the same structure with respect to G and so there exists $g \in G$ such that $Rg = Q$. Hence the result holds.

Suppose that $y \in P_2$, i.e. $S_2 = S_2' \cup \{y\}$, and consider the partition $R = (R_1, R_2, \ldots)$ where

$$R_1 = S_1 \cup \{y\}, \quad R_2 = S_2' \cup \{x_1, x_2\}, \quad R_i = P_i = Q_i \text{ for all } i > 2.$$

The partitions P and R have the same structure with respect to G and both have shape λ. Hence there exists $g \in G$ such that $Pg = R$. Similarly the partitions R and Q have the same structure with respect to H and so there exists $h \in H$ such that $Rh = Q$. Hence the result holds.

If $y \notin P_1 \cup P_2$ then, without loss of generality, it can be assumed that $y \in P_3$, i.e. $P_3 = S_3 \cup \{y\}$ for some $S_3 \subseteq V$. Consider the partitions $R = (R_1, R_2, \ldots)$ where

$$R_1 = S_1 \cup \{y\}, \quad R_2 = S_2 \cup \{x_2\}, \quad R_3 = S_3 \cup \{x_1\},$$
$$R_i = P_i = Q_i \text{ for all } i > 3,$$

and $T = (T_1, T_2, \ldots)$ where

$$T_1 = S_1 \cup \{x_2\}, \quad T_2 = S_2 \cup \{y\}, \quad T_3 = S_3 \cup \{x_1\},$$
$$T_i = P_i = Q_i \text{ for all } i > 3.$$

Note that both partitions have shape λ. The partitions P and R have the same structure with respect to G, hence there exists $g \in G$ such that $Pg = R$. The partitions R and T have the same structure with respect to H, hence there exists $h \in H$ such that $Rh = T$. The partitions T and Q have the same structure with respect to G, hence there exists $g' \in G$ such that $Tg' = Q$. Hence the result holds when $m = 1$.

Assume, as induction hypothesis, that the theorem holds for a given value of m and consider the case when $\sigma' = g_1 h_1 \dots g_{m+1} h_{m+1}$. The above techniques may be repeated, with $y = x_1 g_1 h_1 \dots g_m h_m$, to show that the theorem holds for $m+1$ and so for all values of $m \geq 1$. □

Hence any permutation group G on V has a unique subgroup $\Omega_\lambda(G)$ which is maximal with respect to being orbit-λ-transitive. Moreover, this subgroup is normal in G.

If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a shape of a partition of V then, without loss of generality, it can be assumed that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k .$$

Furthermore, if $\mu = (\mu_1, \dots, \mu_m)$ is the shape of another partition of V then a partial ordering can be defined where μ dominates λ, written $\lambda \trianglelefteq \mu$, if

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$$

for all j (with the convention that $\lambda_i = 0$ for all $i > k$, and similarly for μ). Hence the set of shapes of V forms a lattice.

The following result, a more generalised version of the result of Livingstone and Wagner [**6**], is also true:

Theorem 3. *Let μ dominate λ and suppose that G is orbit-λ-transitive. Then G is orbit μ-transitive.*

Proof. It is enough to prove this in the case where μ covers λ in the partition lattice, since we can then prove the theorem by induction on the length of the chain connecting them. This means that there exist $j < k$ such that

$$\mu_j = \lambda_j + 1, \quad \mu_k = \lambda_k - 1, \quad \mu_i = \lambda_i \text{ for } i \neq j, k.$$

Suppose that G is orbit λ-transitive. Let (S_i) and (T_i) be two orbit-equivalent partitions with $|S_i| = |T_i| = \mu_i$ for all i. We have to show that some element of g carries the first partition to the second. This follows from the Martin–Sagan result [**7**] if G is transitive, so we may suppose not.

Since $\mu_j > \mu_k$, there is an orbit V of G such that $|V \cap S_j| > |V \cap S_k|$. Choose $x \in V \cap S_j$ and let $S_j^* = S_j \setminus \{x\}$, $S_k^* = S_k \cup \{x\}$, and $S_i^* = S_i$ for $i \neq j, k$; construct T^* similarly. Then S^* and T^* are orbit-equivalent partitions of shape λ, and so there exists $g \in G$ carrying S^* to T^*. This element carries $S_i \setminus V$ to $T_i \setminus V$ for all i, so we can assume these sets are equal.

Since $|V \cap S_j| > |V \cap S_k|$, the shape of the partition λ' of V induced by S^* is dominated by the shape μ' of the partition induced by S. Now the stabiliser of all the sets $S_i \setminus V$ is transitive on partitions of V of shape λ'. By Martin and Sagan again, it is transitive on partitions of shape μ', so there is an element h fixing all $S_i \setminus V$ and mapping all $S_i \cap V$ to $T_i \cap V$. So we are finished. □

Corollary 1. *If G is an orbit-t-homogeneous permutation group on a set V, where $|V| \geq 2t - 1$ and $t > 1$ then G is orbit-$(t-1)$-homogeneous.*

This result also shows that, with the earlier notation, an arbitrary permutation group G of degree n induces a lattice of normal subgroups $\Omega_\lambda(G)$ where $\Omega_\lambda(G) \leq \Omega_\mu(G)$ whenever $\lambda \trianglelefteq \mu$. It is clear that if $\lambda = (n)$ or $\lambda = (n-1,1)$ then $\Omega_\lambda(G) = G$ and that if $\lambda = (1,1,\dots,1)$ then $\Omega_\lambda(G) = 1_G$ unless G is the symmetric group (in which case $\Omega_\lambda(G) = G$ for all λ).

For $t \in \{1,2,\dots,\lfloor n/2 \rfloor\}$, let $\Omega_t(G)$ denote $\Omega_\lambda(G)$ where $\lambda = (n-t,t)$. Hence $\Omega_t(G)$ is the maximal subgroup of G that is orbit-t-homogeneous.

Theorem 4. *Suppose G is a permutation group with degree n that acts on a set V with d orbits. If $(\lambda_1, \lambda_2, \dots, \lambda_k)$ is a chain of shapes of V such that $\lambda_{i+1} \trianglelefteq \lambda_i$ for all $1 \leq i \leq k-1$ then*

$$|\{\Omega_{\lambda_i}(G) : 1 \leq i \leq k\}| \leq d+2$$

Proof. Every shape except (n) and $(n-1,1)$ is dominated by $(n-2,2)$. Hence $\Omega_{\lambda_i}(G)$ is orbit-2-homogeneous for all $1 \leq i \leq k$ except, possibly, when $i=1$ and $i=2$. Now, $\Omega_2(G)$ acts primitively on its orbits; so, for each $\lambda_i \trianglelefteq (n-2,2)$, the normal subgroup $\Omega_{\lambda_i}(G)$ must act either transitively or trivially on each $\Omega_2(G)$-orbit. Furthermore, if $\Omega_{\lambda_i}(G)$ acts trivially on a $\Omega_2(G)$-orbit then it acts trivially on all the $\Omega_2(G)$-orbits in the same G-orbit.

Therefore, either $\Omega_{\lambda_{i+1}}(G)$ acts trivially on exactly the same G-orbits as $\Omega_{\lambda_i}(G)$, and so $\Omega_{\lambda_{i+1}}(G) = \Omega_{\lambda_i}(G)$, or there exists at least one G-orbit on which $\Omega_{\lambda_{i+1}}(G)$ acts trivially and $\Omega_{\lambda i}(G)$ does not. If $\Omega_{\lambda_i}(G)$ acts trivially on every G-orbit then $\Omega_{\lambda_i}(G) = 1_G$. Hence the result holds. □

This means that in the case of orbit-t-homogeneous groups things are, in fact, quite restricted.

Corollary 2. *If G is transitive then one of the following holds:*
(a) $\Omega_1(G) = G$, $\Omega_t(G) = 1_G$ for all $1 < t \leq n/2$.
(b) There is a non-trivial normal subgroup $N \trianglelefteq G$ such that $\Omega_1(G) = G$, $\Omega_t(G) = N$ for all $1 < t \leq n/2$.
(c) There is a non-trivial normal subgroup $N \trianglelefteq G$ and an integer $m > 1$ such that $\Omega_1(G) = G$, $\Omega_t(G) = N$ for $1 < t \leq m$, and $\Omega_t(G) = 1_G$ for all $m < t \leq n/2$.

As a series of examples, consider a group H that acts on a set of n points ($n \geq 2$), and the wreath product $G = \mathrm{Wr}(H, C_2) = (H \times H) \cdot C_2$ that acts on a set V of $2n$ points in the natural way.

- If $H \cong C_n$ then $\Omega_1(G) = G$ and $\Omega_t(G) = 1$ for all $1 < t \leq n$.
- If $H \cong S_n$ then $\Omega_1(G) = G$ and $\Omega_t(G) = H \times H$ for all $1 < t \leq n$.
- If H is u-homogeneous but not $(u+1)$-homogeneous, for $1 < u < n$, then $\Omega_1(G) = G$, $\Omega_t(G) = H \times H$ for $2 \leq t \leq u$, and $\Omega_t(G) = 1$ for $u < t \leq n$. Such groups exist only for $u \leq 5$ (by the main result of Livingstone and Wagner and the classification of t-transitive groups).

For intransitive groups, things are not so restricted, as the examples in the following remarks show.

Remark 1. A permutation group G with two orbits V_1 and V_2 is orbit 2-homogeneous if and only if G is 2-homogeneous on each orbit and transitive on

$V_1 \times V_2$ (equivalently, the permutation characters of G on V_1 and V_2 are different). There are many examples of such groups. In particular, both, one, or neither of the actions of G on V_1 and V_2 may be faithful, as the following examples show:

- $\mathrm{PSL}(2,7)$, with orbits of size 7 and 8;
- $\mathrm{P\Gamma L}(2,8)$, with orbits of size 3 and 28;
- the direct product of two 2-homogeneous groups.

REMARK 2. Let G be a group having all orbits of size 2 (say $O_1, \ldots, O_m$). With each $g \in G$, associate the m-tuple $(e_1, \ldots, e_m)$, where $e_i = 0$ or 1 according as g fixes O_i pointwise or not. Then G is orbit t-homogeneous if and only if the set of all these m-tuples is an orthogonal array of strength t, for any $t \le m$. (This means that, given any t coordinates $i_1, \ldots, i_t$, and any t values $\epsilon_1, \ldots, \epsilon_t \in \{0,1\}$, there is a constant number λ of elements $g \in G$ whose associated m-tuple satisfies $e_{i_j} = \epsilon_{i_j}$ for $j = 1, \ldots, t$.)

To prove this, note that for a group the requirement of being an orthogonal array of strength t is equivalent to the formally weaker requirement that, given $i_1, \ldots, i_t$ and $\epsilon_1, \ldots, \epsilon_t$, there is some element of G with the required property (since there will then be $|G|/2^t$ such elements). Now take any two orbit-equivalent t-sets S_1 and S_2. Let $i_1, \ldots, i_s$ be the indices i for which S_1 and S_2 meet the ith orbit in singletons, and put $\epsilon_{i_j} = 0$ if $S_1 \cap O_{i_j} = S_2 \cap O_{i_j}$, $\epsilon_{i_j} = 1$ otherwise. Now the element g guaranteed by the strength-s property of the orthogonal array maps S_1 to S_2. The converse is proved by reversing the argument.

In particular, if G consists of all even permutations fixing the orbits, then it is an orthogonal array of strength $m-1$. This shows that there are orbit t-homogeneous groups with arbitrarily large t.

REMARK 3. If G has all orbits of size 3, then G is orbit t-homogeneous if and only if its (normal) Sylow 3-subgroup is. The criterion for this is almost identical to that in Remark 1, using the alphabet $\{0,1,2\}$. Also, if G has all orbits of size 2 or 3, then G is orbit t-homogeneous if and only if the groups induced on the union of orbits of each size are. We do not give details.

REMARK 4. The situation for orbits of size 4 or more is a bit more complicated. We can give a partial description of the orbit 4-homogeneous groups as follows.

PROPOSITION 1. *Let G be orbit 4-homogeneous of degree at least 8, and let H be the third derived group of G. Then H is a direct product of simple groups taken from the list A_n ($n \ge 5$), M_n ($n = 11, 12, 23, 24$), and $\mathrm{PSL}(2,q)$ ($q = 5, 8, 32$), each factor acting transitively on one G-orbit and fixing all the others pointwise.*

Proof. The 4-homogeneous groups which are not 4-transitive have been classified by Kantor [**4**], and the list of 4-transitive groups follows from the classification of finite simple groups. All of them have simple derived groups in the list in the proposition. Groups of degree at most 4 have derived length at most 3. By inspection, a group on the above list cannot act non-trivially on two different orbits in an orbit-4-homogeneous group. □

There remains some subtlety in the structure of G. For example:

- The Proposition gives no information about orbits of size at most 4. In particular, the examples described in Remarks 2 and 3 are completely invisible from this point of view.

– Any group G lying between a direct product $\prod_{i=1}^{r} S_{n_i}$ of symmetric groups and its derived group $\prod_{i=1}^{r} A_{n_i}$ (with $n_i \geq 5$ for all i) is orbit 4-homogeneous. We can add orbits of length 2 on which $G/\prod A_{n_i}$ acts as in Remark 1.
– The group $\mathrm{P\Gamma L}(2,8)$, acting with orbits of size 3 and 9, is orbit 4-homogeneous. The transitivity on 4-sets containing one point from the orbit of length 3 follows from the 3-homogeneity of $\mathrm{PSL}(2,8)$.

REMARK 5. The above Proposition fails for orbit 3-homogeneous groups. The groups S_6 (with two inequivalent orbits of size 6) and M_{12} (with two inequivalent orbits of size 12) are orbit 3-homogeneous but not orbit 4-homogeneous. Other examples include $(C_2^r)^m \cdot \mathrm{GL}(r,2)$, for $m, r \geq 2$, with m orbits of size 2^r.

REMARK 6. If $G = G_1 \times \ldots \times G_5$, where G_t is t-homogeneous but not $(t+1)$-homogeneous, then $\Omega_t(G) = \Omega_t(G_1) \times G_t \times \ldots \times G_5$ for $2 \leq t \leq 5$.

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