Posets, homomorphisms and homogeneity

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Abstract

Jarik Nešetřil suggested to the first author the investigation of notions of homogeneity for relational structures, where "isomorphism" is replaced by "homomorphism" in the definition. Here we look in detail at what happens for posets. For the strict order, all five generalisations of homogeneity coincide, and we give a characterisation of the countable structures that arise. For the non-strict order, there is an additional class. The "generic poset" plays an important role in the investigation.

Key words: Poset, homomorphism, homogeneous, relational structure

1 Introduction

Among the many mathematical topics to which Jarik Nešetřil has made important contributions are

- *homomorphisms* of relational structures (the book [5] is the standard);
- *homogeneous structures*, which are closely connected with Ramsey classes;
- *posets* (for example, the construction of the generic poset in [6]).

The purpose of this paper is to explore an area of overlap of these three topics. Indeed, Jarik must take further responsibility for this, since he suggested to the first author in 2004 the idea of combining the last two topics. In [2], we were mainly concerned with graphs, but made a preliminary investigation of posets. The results of this paper complete that investigation.

The organisation of the paper is as follows. In the next section, we describe the classical notion of homogeneity due to Fraïssé, and explain how to extend it to homomorphisms, defining six classes of "homomorphism-homogeneous" structures including the classical one. Next we sketch the construction of the countable generic poset given by Hubicka and Nešetřil [6]. In the main part of the paper we give the main results. For posets with strict order, the five non-classical classes of homomorphism-homogeneous structures are all equivalent, while for non-strict order they fall into just two types. We give characterisations of these structures and some examples, but stop short of a completely explicit description.

2 Homogeneity

A countable relational structure M belonging to a class $\mathcal{P}$ is

- *universal* if every finite or countable structure in $\mathcal{P}$ is embeddable in M (as induced substructure);
- *homogeneous* if every isomorphism between finite substructures of M can be extended to an automorphism of M (an isomorphism $M \to M$).

The *age* of a relational structure M is the class $\mathcal{C}$ of all finite structures embeddable in M.

In about 1950, Fraïssé gave a necessary and sufficient condition on a class $\mathcal{C}$ of finite structures for it to be the age of a countable homogeneous structure M.

The key part of this condition is the *amalgamation property*: two structures in $\mathcal{C}$ with isomorphic substructures can be "glued together" so that the substructures are identified, inside a larger structure in $\mathcal{C}$. We allow the isomorphic substructures to be empty, so that the amalgamation property includes the joint embedding property. The remaining conditions are easier: $\mathcal{C}$ should be closed under isomorphisms and hereditary (closed under induced substructures), and should contain countably many members up to isomorphism.

Moreover, if $\mathcal{C}$ satisfies Fraïssé's conditions, then M is unique up to isomorphism; we call $\mathcal{C}$ a *Fraïssé class* and M its *Fraïssé limit.*

Homogeneous structures have a variety of uses. There are connections with zero-one laws; they have been used for constructions in model theory and permutation group theory; and recently it has been shown that their automorphism groups may possess very strong amenability properties [7]. Nešetřil (see [6]) observed a connection with Ramsey theory: if the age of a structure with an explicit or implicit total order is a Ramsey class, then it is a Fraïssé class. This observation led him to his recent proof that finite metric spaces form a Ramsey class [9], and to a programme to determine the Ramsey classes [10].

3 The generic poset

The class of all finite graphs is obviously a Fraïssé class. Its Fraïssé limit R is the celebrated *countable random graph* or *Rado graph*, whose properties are surveyed in [1]. In this paper we are concerned with the analogous poset, the so-called *generic poset*, which we denote by U.

A *partially ordered set* or *poset*, P, is a relational structure $(\Omega, \leq)$ such that the relation $\leq$ is reflexive, antisymmetric, and transitive on the set Ω. Equivalently, we may consider a poset P as a set Ω with a strict order, $<$, which is irreflexive and transitive. All posets considered will be finite or countably infinite.

The generic poset U is the Fraïssé limit of the class of finite posets. It has the properties

- U is the unique countable homogeneous universal poset;
- U is the generic countable poset: that is, if the set of all posets on the set $\mathbb{N}$ is made into a metric space by defining the distance between two posets to be $1/2^n$ if the largest initial interval of $\mathbb{N}$ on which they agree is $[0, n]$, then the posets isomorphic to U form a residual set.

The generic poset U is characterized by the property that, for any finite disjoint sets A, B, C with $A < B$ and, for all $a \in A, b \in B, c \in C$, we have $c \not< a$ and $b \not< c$, there is a point z with $A < z < B$ and $z \parallel C$. Here we write $a \parallel b$ to mean that a and b are incomparable, and $z < B$ if $z < b$ for all $b \in B$, with similar notation for other relations between points and subsets.

Schmerl [11] classified all the countable homogeneous posets:

Theorem 1 (Schmerl) *A countable poset is homogeneous if and only if it is one of the following four types:*

- A_n*, an antichain on* $n \in \mathbb{N}^*$ *points (where* $\mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$*);*
- B_n*, the disjoint union of* $n \in \mathbb{N}^*$ *copies of* $(\mathbb{Q}, \leq)$*;*
- C_n*, a copy of* B_n *with additional ordering between* $\mathbb{Q}_i$ *and* $\mathbb{Q}_j$ *for any* i, j*, given by* $(i, x) < (j, y)$ *if and only if* $x < y$*;*
- U*, the generic countable poset.*

Unlike the random graph R, there is no known simple and explicit construction of U. The nicest construction is due to Hubicka and Nešetřil; we give a very brief sketch.

Take a countable model of set theory with a single atom $\diamondsuit$. Now let M be any set not containing $\diamondsuit$. Put

$M_L = \{A \in M : \diamondsuit \notin A\}$,
$M_R = \{B \setminus \{\diamondsuit\} : \diamondsuit \in B \in M\}$.

Then neither M_L nor M_R contains $\diamondsuit$.

In the other direction, given two sets P, Q whose elements don't contain $\diamondsuit$, let $(P \mid Q) = P \cup \{B \cup \{\diamondsuit\} : B \in Q\}$. Then $(P \mid Q)$ doesn't contain $\diamondsuit$. Moreover, for any set M not containing $\diamondsuit$, we have $M = (M_L \mid M_R)$.

Note that any set not containing $\diamondsuit$ can be represented in terms of sets not involving $\diamondsuit$ by means of the operation $(. \mid .)$ For example, $\{\emptyset, \{\diamondsuit\}\}$ is $(\{\emptyset\} \mid \{\emptyset\})$.

Let $\mathcal{P}$ be the collection of the sets M not containing $\diamondsuit$ defined by the following recursive properties:

Correctness: $M_L \cup M_R \subseteq \mathcal{P}$ and $M_L \cap M_R = \emptyset$.
Ordering: For all $A \in M_L$ and $B \in M_R$, we have

$$(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset.$$

Completeness: $A_L \subseteq M_L$ for all $A \in M_L$, and $B_R \subseteq M_R$ for all $B \in M_R$.

Now for $M, N \in \mathcal{P}$, we put $M \leq N$ if

$$(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset.$$

Theorem 2 $(\mathcal{P}, \leq)$ *is isomorphic to the generic poset U.*

We remark that this construction is analogous both to the construction of R as a countable model of Zermelo–Fraenkel set theory with symmetrised membership as adjacency, and to Conway's construction of numbers [3,8].

4 Homomorphisms and homogeneity

Let A and B be relational structures over the same relational language. A *homomorphism* is a map $f : A \to B$ between the sets which maps every instance of a relation in A to an instance of the same relation in B. We refer to [5] for more details.

For posets, we may use either the strict or the non-strict order; these give different notions of homomorphism, since a homomorphism of the non-strict order can map a_1 and a_2 to the same point if $a_1 < a_2$ (this is impossible for the strict order).

A *monomorphism* is a one-to-one homomorphism. Note that, for posets, the monomorphisms of the strict and non-strict order are the same. An *isomorphism* is a bijective homomorphism whose inverse is also a homomorphism.

A relational structure S is homogeneous if any isomorphism between finite substructures of S can be extended to an automorphism of S. We look at the new classes of relational structures that arise when this definition of homogeneity is changed slightly, by replacing 'isomorphism' (and 'automorphism') by 'homomorphism' or 'monomorphism'. We say that a structure S belongs to the class XY if every x-morphism from a finite substructure of S into S extends to a y-morphism from S to S; where (X, x) and (Y, y) can be (I,iso), (M,mono), or (H,homo). The classes that arise are IH, MH, HH, IM, MM, and II. (It is not reasonable to extend a map to one satisfying a stronger condition!) Note that the class II is the class of homogeneous structures.

We have the following hierarchy picture for these classes (where class A is written below B to mean that A is a subclass of B):

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              IH
            /    \
        IM          MH
      /    \      /    \
    II        MM          HH
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Now we can state part of our results for posets.

Theorem 3 *(a) For posets with strict order, the classes* IM, IH, MM, MH, HH *coincide, and properly contain the class* II.

(b) For posets with non-strict order, the classes IH, MH, HH *coincide and properly contain the classes* IM *and* MM *(which are the same as for strict order), in turn properly containing the class* II.

We give also a description of the posets in these classes: see Propositions 15 and 25.

5 Posets

The set of elements of a poset is denoted by Ω. A *chain* is a poset P such that either $x \leq y$ or $x > y$ for each pair $x, y \in \Omega$. An *antichain* is a poset P such that $x \parallel y$ for each pair $x, y \in \Omega$ with $x \neq y$. We will always consider chains and antichains to be nonempty. A chain (antichain) consisting of just one element is called a *trivial* chain (antichain). So a trivial chain is the same as a trivial antichain: precisely an isolated point.

A poset P is *dense* if for all $x, y \in \Omega$ with $x < y$, there exists $z \in \Omega$ such that $x < z < y$. *Maximal* and *minimal* elements $m_1, m_2 \in \Omega$ are those such that there is no $z \in \Omega$ with $m_1 < z$, or with $z < m_2$, respectively. For $X \subset \Omega$ and $z \in \Omega$, if $z \geq X$ ($z \leq X$), then z is an *upper (lower) bound* of X.

The following result appears in [2]:

Proposition 4 *(a) A countable poset P is an extension of U (i.e. a poset P on the same set such that if $x \leq_U y$ then $x \leq_P y$) if and only if it has the following property:*
(†) for any two finite subsets $A, B \subset \Omega$ with $A < B$, there exists $z \in \Omega$ such that $A < z < B$.
(b) Any extension of U is HH *and* MM.

Proposition 5 *If a countable poset is* II*, then it is also* HH *and* MM.

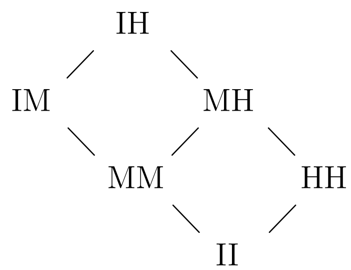
PROOF. Clearly each A_n is HH and MM (since there is no ordering to be preserved in an antichain).

The fact that B_n is HH and MM for each $n \in \mathbb{N}^*$ follows from the fact that $(\mathbb{Q}, \leq)$ is HH and MM (since it is dense and without endpoints).

For each $n \in \mathbb{N}^*$ the property (†) holds for the poset C_n (by the fact that $(\mathbb{Q}, \leq)$ is dense), hence by Proposition 4 each is an extension of U and so is HH and MM.

Trivially U is an extension of itself, so is HH and MM by Proposition 4.

Now we can make a first refinement to the hierarchy picture for the classes, since by Proposition 5, the class II is a subclass of the classes HH and MM. So the picture becomes



Investigation of MH posets began in [2]. We will continue this work, considering the more general class IH.

Lemma 6 *For any poset P, the following conditions are equivalent:*

(i) every 2-element antichain has an upper (lower) bound
(ii) every finite antichain has an upper (lower) bound
(iii) every finite subset has an upper (lower) bound

We leave the proof as an exercise for the reader.

Proposition 7 *Let P be a countable* IH *poset which is not an antichain.*

(a) P is dense and has no maximal or minimal elements (and so $|\Omega|$ must be infinite).
(b) If P is disconnected, then it is a disjoint union of incomparable chains each isomorphic to $(\mathbb{Q}, \leq)$, i.e. B_n for some $n > 1$.
(c) If there is a 2-element antichain in P with an upper (lower) bound, then any finite subset has an upper (lower) bound.

PROOF. (a) Since P is not an antichain, there exist $a, b \in \Omega$ with $a < b$. Then extending the isomorphism $a \mapsto x$ for any x to a homomorphism, we see that there is an element above x (the image of b). Similarly there is an element below x. Hence P does not have maximal or minimal elements.

Also, now there is a 3-element chain $a < b < c$ in P. So for any $x < y$, extending the isomorphism $a \mapsto x,\ c \mapsto y$ to a homomorphism we see that there is an element $z \in \Omega$ (the image of b) with $x < z < y$. Hence P is dense.

Clearly this means that $|\Omega|$ must be infinite.

(b) Suppose P has a component which is not a chain. Then it has a 2-element antichain with either an upper or lower bound. By the IH property, every 2-element antichain has a bound, so there is only one connected component.

(c) As in (b), if some 2-element antichain has an upper bound, then by the IH property, every 2-element antichain has an upper bound. Then by Lemma 6 every finite subset has an upper bound. Similarly for lower bounds.

From now on we assume that P is connected, in particular P is not an antichain, and that Ω is countably infinite.

Proposition 8 *Let P be an* IH *poset. Then for any finite subset (which may be empty) $Q \subset P$, $P_{(<Q)} := \{z \in P : z < Q\}$ has no maximal elements, and $P_{(>Q)} := \{z \in P : z > Q\}$ has no minimal elements.*

PROOF. Suppose there exists some finite subset $Q \subset P$ for which $P_{(<Q)}$ has a maximal element, m. Now there exists $c \in P_{(<Q)}$ with $c < m$, since P does not have minimal elements by Proposition 7(a). Extending the isomorphism which fixes Q and sends $c \mapsto m$ to a homomorphism, we see that there exists $l \in P$ (the image of m) such that $m < l < Q$. But this contradicts the definition of m as a maximal element of $P_{(<Q)}$. Hence $P_{(<Q)}$ cannot have a maximal element.

Similarly, for any finite subset $Q \subset P$, $P_{(>Q)}$ cannot have a minimal element.

Lemma 9 *Let P be a countable poset. If for any finite subset $Q \subset P$, $P_{(<Q)}$ has no maximal elements, then P is dense and has no maximal elements. If for any finite subset $Q \subset P$, $P_{(>Q)}$ has no minimal elements, then P is dense and has no minimal elements.*

PROOF. Suppose that the first condition holds, i.e. for any finite subset $Q \subset P$, $P_{(<Q)}$ has no maximal elements. Take any $x, y \in P$ with $x < y$, and consider $Q = \{y\}$. Then $P_{(<Q)} = P_{(<y)}$ has no maximal elements, in particular, x is not a maximal element of this set. So there exists $z \in P_{(<y)}$ with $x < z$, that is there exists $z \in P$ with $x < z < y$. So P is dense. Similarly for the second condition.

Consider $Q = \emptyset$. Then by the first condition, $P_{(<Q)} = P_{(<\emptyset)} = P$ has no maximal elements, and by the second condition, $P_{(>Q)} = P_{(>\emptyset)} = P$ has no minimal elements.

A *tree* is a connected poset $(\Omega, \leq)$ such that for each $a \in \Omega$ the set $\{z \in \Omega : z < a\}$ is a chain. Note that an equivalent definition for a tree is a poset such that no 2-element antichain has an upper bound, but all have lower bounds. So a tree is precisely the case of a poset such that all finite subsets have lower bounds, but no nontrivial antichain has an upper bound. An *inverted tree* (or *inversion of a tree*) will be an upside-down tree, i.e. a poset such that all finite subsets have upper bounds but no nontrivial antichain has a lower bound.

Proposition 10 *Let P be a tree with no minimum element such that for each finite $Q \subset P$, $P_{(<Q)}$ has no maximal elements. Then for each $Q \subset P$, $P_{(>Q)}$ has no minimal elements. Moreover, P is* HH *and* MM*.*

PROOF. Firstly, note that since P is a tree, for each finite nonempty $Q \subset P$, $P_{(<Q)}$ is a nonempty chain. So in this case the condition says that $P_{(<Q)}$ has no maximum element. In the case where Q is empty, it says that P does not have maximal elements.

If we show that P is HH and MM, then certainly it is IH. Then the condition that for each $Q \subset P$, $P_{(<Q)}$ has no minimal elements follows from Proposition 8. However we may also show it directly.

If $Q = \emptyset$, then $P_{(>Q)} = P$, and we have stated that the tree does not have a minimum element. Now, for some finite nonempty $Q \subset P$, if $P_{(>Q)}$ is nonempty, then Q is a chain (since in a tree no antichain has an upper bound). Since Q is a finite chain, it has a maximum element, say x. Suppose $P_{(>Q)}$ has a minimal element, say m. Now consider $P_{(<m)}$, x is a maximal element of this set (else there exists y with $x < y < m$, so then m was not a minimal element of $P_{(>Q)}$). But this contradicts the condition that for each finite nonempty $Q \subset P$, $P_{(<Q)}$ has no maximum element. So $P_{(>Q)}$ did not have a minimal element after all. If for some finite nonempty $Q \subset P$, $P_{(>Q)}$ is empty, then vacuously it has no minimal elements. So indeed for each finite $Q \subset P$, $P_{(>Q)}$ has no minimal elements.

Now let $f : A \to B$ be a homomorphism between finite subsets $A, B \subset \Omega$. Pick some $x \in \Omega \setminus A$. We wish to extend map f for x.

Let $A_{(<x)} := \{z \in A : z < x\}$, $A_{(>x)} := \{z \in A : x < z\}$, $B_{(<)} := f(A_{(<x)})$, $B_{(>)} := f(A_{(>x)})$. Note that $B_{(<)} < B_{(>)}$ since clearly $A_{(<x)} < A_{(>x)}$ and f preserves order. Also for all $z \in A \setminus (A_{(<x)} \cup A_{(>x)})$, $z \parallel x$.

We wish to find $y \in \Omega \setminus B$ such that $B_{(<)} < y < B_{(>)}$.

Firstly, suppose that $B_{(<)}, B_{(>)}$ are both nonempty. Observe $A_{(<x)}$ is a finite chain, and since f is a homomorphism, so is $B_{(<)}$. Let m be the greatest element of $B_{(<)}$. Since $B_{(>)}$ is finite we know that $P_{(<B_{(>)})}$ does not have maximal elements. Clearly $B_{(<)} \subset P_{(<B_{(>)})}$, so in particular, m is not a maximal element of $P_{(<B_{(>)})}$. So we can find $y \in P_{(<B_{(>)})}$ with $y > m$. Then $y \notin B$ and $B_{(<)} < y < B_{(>)}$ as required.

Now define $f(x) = y$. Then clearly this will be order-preserving, since if we consider $z \in A$,

- if $z < x$, then $f(z) \in B_{(<)}$, so $f(z) < y = f(x)$ since $y > B_{(<)}$.
- if $z > x$, then $f(z) \in B_{(>)}$, so $f(x) = y < f(z)$ since $y < B_{(>)}$.
- if $z \parallel x$, then there are no new relations to preserve.

So the extension is a homomorphism.

If $B_{(<)}, B_{(>)}$ are both empty, then it must be that $A_{(<x)}, A_{(>x)}$ were too. Then for all $z \in A, z \parallel x$, so there are no new relations to preserve, so we may pick any $y \in \Omega \setminus B$ to map x to. Then the extension is a homomorphism.

Suppose $B_{(<)}$ is empty, but $B_{(>)}$ is not. Since P is a tree, all finite subsets have lower bounds. So there exists $y \in P$ with $y \leq B_{(>)}$. Since P does not have a minimum element, we may in fact assume that $y < B_{(>)}$, and so $y \notin B$. Mapping x to y all relations are preserved, so the extension is a homomorphism.

Suppose $B_{(>)}$ is empty, but $B_{(<)}$ is not. As before, $A_{(<x)}, B_{(<)}$ are finite chains, and let m be the greatest element of $B_{(<)}$. Since P does not have maximal elements (by Lemma 9), there exists $y \in P$ with $y > m$, i.e. $y \in \Omega \setminus B$ with $y > B_{(<)}$. Mapping x to y all relations are preserved, so the extension is a homomorphism.

So we have shown how to extend any homomorphism by one point to get another homomorphism. Then since P is countable, we can extend f in countably many steps to get a homomorphism $\psi : P \to P$ which extends f. So P is HH. Also, if f was one-to-one, then so is the extension (since we always ensured $y \in \Omega \setminus B$). So P is MM.

The dual of this result is proved in the same way:

Proposition 11 *Let P be an inverted tree with no maximum element such that for each finite $Q \subset P$, $P_{(>Q)} := \{z \in P : z > Q\}$ has no minimal elements. Then for each $Q \subset P$, $P_{(<Q)}$ has no maximal elements. Moreover, P is* HH *and* MM.

We now consider posets which are not trees or inverted trees, i.e. posets containing some nontrivial antichains with upper bounds, and some with lower bounds. Note that by Proposition 7(c), to be an IH poset in fact every finite subset must have upper and lower bounds.

We will call a 4-tuple (a_1, a_2, b_1, b_2) of elements from Ω a *×-set* (*"cross-set"*), if $a_1 \parallel a_2$, $b_1 \parallel b_2$ and $\{a_1, a_2\} < \{b_1, b_2\}$. Now for a ×-set (a_1, a_2, b_1, b_2), we will call any element $c \in \Omega$ with $\{a_1, a_2\} < c < \{b_1, b_2\}$ a *midpoint* of the ×-set.

Lemma 12 *If P is an* IH *poset, then either*

(a) every ×-set has a midpoint; or
(b) no ×-set has a midpoint.

PROOF. Suppose there exists a ×-set in P (otherwise vacuously (a) holds). So suppose there exists a ×-set (a_1, a_2, b_1, b_2) with a midpoint c, otherwise (b) holds. Now say (x_1, x_2, y_1, y_2) is any ×-set. Extending the isomorphism $a_1 \mapsto x_1,\ a_2 \mapsto x_2,\ b_1 \mapsto y_1,\ b_2 \mapsto y_2$ to a homomorphism, we find $z \in \Omega$ with $\{x_1, x_2\} < z < \{y_1, y_2\}$ (z is the image of c). So every ×-set has a midpoint, i.e. (a) holds.

Proposition 13 *Poset P is an extension of U if and only if*

(i) for each finite $Q \subset P$, $P_{(<Q)}$ is nonempty and has no maximal elements and $P_{(>Q)}$ is nonempty and has no minimal elements;

(ii) every $\times$-set of P has a midpoint.

PROOF. ($\Rightarrow$): If P is an extension of U, then P is HH and MM by Proposition 4(b), so P is IH by definition. Then by Proposition 8, for each finite $Q \subset P$, $P_{(<Q)}$ has no maximal elements and $P_{(>Q)}$ has no minimal elements. It remains to show that these sets are nonempty. By Proposition 4(a), P satisfies ($\dagger$), i.e. for any two finite subsets $A, B \subset \Omega$ with $A < B$, there exists $z \in \Omega$ such that $A < z < B$. Now consider $A = \emptyset$, $B = Q$; by ($\dagger$) there exists z such that $z < Q$, so $P_{(<Q)}$ is nonempty. Similarly, consider $A = Q$, $B = \emptyset$; by ($\dagger$) there exists z such that $Q < z$, so $P_{(>Q)}$ is nonempty. Hence (i) holds.

Let P be an extension of U, and let (a_1, a_2, b_1, b_2) be a $\times$-set of P. Then we have $A = \{a_1, a_2\},\ B = \{b_1, b_2\}$ with $A < B$. By Proposition 4(a), P satisfies ($\dagger$), so there exists $z \in \Omega$ such that $A < z < B$. Then z is a midpoint of the $\times$-set. So (ii) holds.

($\Leftarrow$): We assume that (i) and (ii) hold, and must show that if we have finite $A, B \subset P$ with $A < B$, then there exists z such that $A < z < B$ (we will call this property ($\dagger'$)).

The proof is by induction on the size of $|A| + |B|$. The induction starts with $A = B = \emptyset$, for which ($\dagger'$) is true since this is just the fact that $P \neq \emptyset$. For the induction step, assume ($\dagger'$) holds for any A^*, B^* with $|A^*| + |B^*| < |A| + |B|$.

We claim that we can assume A, B are antichains and $|A|, |B| \geq 2$. For if $A = \emptyset$, and $B \neq \emptyset$, then by (i), $P_{(<B)} \neq \emptyset$; so we may simply choose $z \in P_{<B}$. Also, if $A = \{a\}$, then $a \in P_{<B}$; and by (i), there exists $z \in P_{<B}$ with $a < z$. Similarly, $|B| \geq 2$. Lastly, if we had $a_1, a_2 \in A$ with $a_1 < a_2$, then we could choose z with $A \setminus \{a_1\} < z < B$; and then $a_1 < z$ also. This completes the proof of the claim (and covers the case that $\times$-sets do not exist).

Now choose $a_1, a_2 \in A,\ b_1, b_2 \in B$. Then (a_1, a_2, b_1, b_2) is a $\times$-set, and by (ii) it has a midpoint, say c. We cannot have $c \leq a$ for some $a \in A$, else $a_1 < c \leq a$ contradicting the fact that A is an antichain. Similarly we cannot have $b \leq c$ for some $b \in B$.

Then $A = A' \cup A''$, where $A' < c,\ A'' \parallel c$; and $B = B' \cup B''$, where $c < B',\ c \parallel B''$. Note that $|A'|, |B'| \geq 2$.

We have $A' < \{c\} \cup B''$; and by induction there exists z_1 with $A' < z_1 < \{c\} \cup B''$. Similarly, there exists z_2 with $\{c\} \cup A'' < z_2 < B'$. Now $\{z_1\} \cup A'' <$

$\{z_2\} \cup B''$, and by induction there exists z with $\{z_1\} \cup A'' < z < \{z_2\} \cup B''$. We also have $A' < z_1 < z < z_2 < B'$, so indeed $A < z < B$ as required. So $(\dagger')$ holds for A, B.

Proposition 14 *If P is a poset satisfying Proposition 13(i) and Lemma 12(b), then P is* HH *and* MM.

PROOF. Let P be a poset such that for each finite $Q \subset P$, $P_{(<Q)}$ is nonempty and has no maximal elements and $P_{(>Q)}$ is nonempty and has no minimal elements, and suppose no $\times$-set of P has a midpoint. We may assume that P does have $\times$-sets, else vacuously (a) from Lemma 12 also holds, so by Proposition 13, P is HH and MM.

Let $f : A \to B$ be a homomorphism between finite subsets $A, B \subset \Omega$. Pick some $x \in \Omega \setminus A$. We wish to extend map f for x.

Let $A_{(<x)} := \{z \in A : z < x\}$, $A_{(>x)} := \{z \in A : x < z\}$, $B_{(<)} := f(A_{(<x)})$, $B_{(>)} := f(A_{(>x)})$. Note that $B_{(<)} < B_{(>)}$ since clearly $A_{(<x)} < A_{(>x)}$ and f preserves order. Also for all $z \in A \setminus (A_{(<x)} \cup A_{(>x)})$, $z \parallel x$.

We wish to find $y \in \Omega \setminus B$ such that $B_{(<)} < y < B_{(>)}$.

Firstly, assume that $B_{(<)}, B_{(>)}$ are both nonempty. Note that either $A_{(<x)}$ or $A_{(>x)}$ must be a chain. Otherwise, we can find 2-element antichains in both sets, forming a $\times$-set with midpoint x, contradicting the fact that no $\times$-set in P has a midpoint. Without loss of generality, suppose $A_{(<x)}$ is a chain, then so is $B_{(<)}$, since f is order-preserving. Let m be the greatest element of the finite chain $B_{(<)}$. Since $B_{(>)}$ is finite, by (i) we know that $P_{(<B_{(>)})}$ does not have maximal elements. Clearly $B_{(<)} \subset P_{(<B_{(>)})}$, so in particular, m is not a maximal element of $P_{(<B_{(>)})}$. So we can find $y \in P_{(<B_{(>)})}$ with and $y > m$. Then $y \notin B$ and $B_{(<)} < y < B_{(>)}$ as required.

Now define $f(x) = y$. Then clearly this will be order-preserving, since if we consider $z \in A$,

- if $z < x$, then $f(z) \in B_{(<)}$, so $f(z) < y = f(x)$ since $y > B_{(<)}$.
- if $z > x$, then $f(z) \in B_{(>)}$, so $f(x) = y < f(z)$ since $y < B_{(>)}$.
- if $z \parallel x$, then there are no new relations to preserve.

So the extension is a homomorphism.

If $B_{(<)}, B_{(>)}$ are both empty, then it must be that $A_{(<x)}, A_{(>x)}$ were too. So since for all $z \in A, z \parallel x$, there are no new relations to preserve, so we may pick any $y \in P \setminus B$ to map x to. Then the extension is a homomorphism.

Now suppose that exactly one of $B_{(<)}, B_{(>)}$ is empty, without loss of generality say $B_{(<)} = \emptyset$. Then also $A_{(<x)} = \emptyset$. Since $B_{(>)}$ is finite, by (i) $P_{(<B_{(>)})}$ is nonempty. So there exists $y \in P_{(<B_{(>)})}$ with $y < B_{(>)}$. Mapping x to y all relations are preserved, so the extension is a homomorphism.

Then since P is countable, we can extend f in countably many steps to get a homomorphism $\psi : P \to P$ which extends f. So P is HH. Also, if f was one-to-one, then so is the extension (since we ensured $y \in \Omega \setminus B$). So P is MM.

Proposition 15 *A countable poset P is* IH *if and only if it is one of the following:*

(1) an antichain on $n \in \mathbb{N}^*$ points;
(2) a disjoint union of $n \in \mathbb{N}^*$ copies of $(\mathbb{Q}, \leq)$;
(3) a tree with no minimum element such that for all finite $Q \subset P$, $P_{(<Q)}$ has no maximal elements (or the inversion of such a tree);
(4) an extension of U;
(5) a poset such that for all finite $Q \subset P$, $P_{(<Q)}$ is nonempty and has no maximal elements, and $P_{(>Q)}$ is nonempty and has no minimal elements, and no $\times$-set has a midpoint.

PROOF. ($\Rightarrow$): Suppose P is IH. Firstly note that by Proposition 8, for all finite $Q \subset P$, $P_{(<Q)}$ has no maximal elements and $P_{(>Q)}$ has no minimal elements, we will call this property ($*$).

If P is not an antichain (else (1) holds) but is disconnected, then by Proposition 7(b) P is a disjoint union of $n \in \mathbb{N}^*$, copies of $(\mathbb{Q}, \leq)$ ($n > 1$), so (2) holds.

Otherwise, assume P is connected. If P is a nontrivial chain, then by Proposition 7(a) it is dense and has no maximal or minimal elements. So P is a copy of $(\mathbb{Q}, \leq)$, and so again (2) holds.

So now assume P is not a chain. Then there exists a 2-element antichain, with either an upper or lower bound, say it has a lower bound. Now by Proposition 7(c), any finite subset of P has a lower bound. By Proposition 7(c), either every 2-element antichain of P has an upper bound, or none does. If no 2-element antichain of P has an upper bound, then P is a tree. By Lemma 9 and ($*$), P has no minimum element, and for all finite $Q \subset P$, $P_{(<Q)}$ has no maximal elements, so (3) holds.

If we instead had that every finite subset has an upper bound, but no antichain has a lower bound, then P is an inverted tree. Then by Lemma 9 and ($*$), P has no maximum element, and for all finite $Q \subset P$, $P_{(>Q)}$ has no minimal elements. So P is the inversion of an IH tree, and so again (3) holds.

Otherwise, every 2-element antichain of P has upper and lower bounds, so by Proposition 7(c) every finite subset of P has upper and lower bounds. That is, for all finite $Q \subset P$, $P_{(<Q)}$ and $P_{(>Q)}$ are both nonempty.

By Lemma 12, if P is IH, then either

(a) every $\times$-set has a midpoint; or
(b) no $\times$-set has a midpoint.

Suppose (a) holds. Then (i) and (ii) of Proposition 13 are satisfied, and so P is an extension of U, so (4) holds.

Otherwise (b) holds, so then (5) holds.

($\Leftarrow$): Follows from Theorem 1 and Propositions 10, 4 and 14, and the fact that if P is II or HH, then it is IH.

Corollary 16 *The classes of countable posets* IH*,* MH*,* HH*,* IM*,* MM*, are all equal.*

PROOF. If P is IH, then by Proposition 15 it is of type (1), (2), (3), (4) or (5). If it is type (1) or (2), then by Theorem 1, P is II, and then by Proposition 5, P is HH and MM. If it is type (3), then by Proposition 10, P is HH and MM. If it is type (4), then by Proposition 4, P is HH and MM. If it is type (5), then by Proposition 14, P is HH and MM. So we have in fact shown that if P is IH, then it is also HH and MM.

By definition, if P is HH, then P is MH, and then P is IH. Similarly if P is MM, then P is IM (and MH), and then P is IH.

Thus for countable posets we have the picture:

$$\begin{array}{c} \text{IH} = \text{MH} = \text{HH} = \text{IM} = \text{MM} \\ | \\ \text{II} \end{array}$$

The class IH is clearly bigger than the class II. In fact it is much bigger! While the class II is countably infinite, the class IH is uncountably infinite.

Proposition 17 *There are uncountably many countable* IH *posets.*

PROOF. Construct an IH poset as follows. Take a discrete tree T; for each pair (x, y) in T such that y covers x, add a copy of the open rational interval

$(0, 1)$ between x and y; and delete the points of T. The resulting tree is IH. Since there are uncountably many discrete trees without divalent vertices, and for different such discrete trees the IH posets constructed will be different, we have uncountably many IH posets.

6 Posets with non-strict order

We now consider *non-strict-order-preserving* maps of posets, that is, maps f such that if $x \leq y$ then $f(x) \leq f(y)$.

To distinguish with the previous strict-order-preserving maps, we will use '$<$' and '$\leq$' prefixes. e.g. we refer to $<$-order-preserving maps and $\leq$-order-preserving maps; $<$-homomorphisms and $\leq$-homomorphisms. For the classes we use subscripts e.g. $\mathrm{IH}_<$ and $\mathrm{IH}_\leq$.

Lemma 18 *A map between posets is a $\leq$-monomorphism if and only if it is a $<$-monomorphism. In particular it is a $\leq$-isomorphism if and only if it is a $<$-isomorphism.*

So the class of posets $\mathrm{II}_\leq$ is exactly the class $\mathrm{II}_<$, and also $\mathrm{IM}_\leq \equiv \mathrm{IM}_<$, and $\mathrm{MM}_\leq \equiv \mathrm{MM}_<$. Since we have already determined $\mathrm{II}_<$, $\mathrm{IM}_<$, and $\mathrm{MM}_<$, we are just left with determining $\mathrm{IH}_\leq$, $\mathrm{MH}_\leq$ and $\mathrm{HH}_\leq$.

Proposition 19 *Let P be an $\mathrm{IH}_\leq$ poset.*

(a) If P is disconnected, then it is a disjoint union of incomparable countable chains (possibly of different lengths, including trivial chains i.e. isolated points).
(b) If there is a 2-element antichain in P with an upper (lower) bound, then any finite subset of P has an upper (lower) bound.

PROOF. Follows exactly as for $\mathrm{IH}_<$ posets, Proposition 7(b) and (c).

Proposition 20 *Let P be a disjoint union of $n \in \mathbb{N}^*$ incomparable countable chains. Then P is $\mathrm{HH}_\leq$.*

Remark. In contrast to the $\mathrm{IH}_<$ case, $\mathrm{IH}_\leq$ posets are not required to be dense. So these chains could be finite, or any of the uncountably many nonisomorphic countably infinite chains.

PROOF. We could proceed as in previous proofs, and show how to extend each finite homomorphism by one point to get another homomorphism. Then by induction we know that after countably many steps we would get a map of the whole of P. However in this case it is possible to define the full extension map explicitly, and since this is quite nice to see, this is how we shall proceed.

Let $P = \left(\bigcup_{1 \leq \alpha \leq k} \mathcal{C}_\alpha \right) \cup \left(\bigcup_{k+1 \leq \alpha \leq k+l} p_\alpha \right)$, where each $\mathcal{C}_\alpha$ is a nontrivial countable chain, and each p_α is an isolated point (trivial chain). We will call these chains and isolated points the *components* of P. The notation is assumed to be such that the components are incomparable with each other.

Let $\phi : A \to B$ be a $\leq$-homomorphism between finite subsets of P. We will define an explicit $\leq$-homomorphism $\psi : P \to P$ which extends ϕ (and in fact maps onto B).

Note that A is a disjoint union of incomparable finite chains and isolated points, say $A = \left(\bigcup_{1 \leq i \leq n} C_i \right) \cup \left(\bigcup_{n+1 \leq i \leq n+m} x_i \right)$, where for each $i \in [n]$, C_i is the finite chain $a_{i,1} < a_{i,2} < \ldots < a_{i,k_i}$ where $k_i \geq 2$; and all the chains C_i and points x_i are incomparable with each other. (We may also observe $n \leq k,\ m \leq k + l - n$).

First we split up parts of P into intervals, I, defined by A, as follows. For each $i \in [n]$ let $(-, a_{i,2}) := \{z \in P : z < a_{i,2}\}$; $[a_{i,j}, a_{i,j+1}) := \{z \in P : a_{i,j} \leq z < a_{i,j+1}\}$ for $2 \leq j \leq k_i - 1$; $[a_{i,k_i}, -) := \{z \in P : z \geq a_{i,k_i}\}$. For each $n < i < n + m$ let $\mathcal{X}_i := \{z \in P : z \leq x_i \text{ or } z > x_i\}$.

Note that for each I, $|I \cap A| = 1$. Also for any intervals I_1, I_2, we have $I_1 \cap I_2 = \emptyset$. For each $i \in [n]$, $(-, a_{i,2}) \cup [a_{i,2}, a_{i,3}) \cup \ldots \cup [a_{i,k_i-1}, a_{i,k_i}) \cup [a_{i,k_i}, -) = \mathcal{C}_\alpha$, where $C_i \subset \mathcal{C}_\alpha$. For each $n < i < n + m$, $\mathcal{X}_i$ is a whole component of P (the component containing x_i), it could be a nontrivial chain or an isolated point.

Choose some $a' \in A$ arbitrarily. Now define ψ by

$$\psi(z) = \begin{cases} \phi(a) & \text{if } z \in I, \text{ where } \{a\} = I \cap A \\ \phi(a') & \text{if } z \in P \setminus (\bigcup I). \end{cases}$$

Clearly ψ extends ϕ, and ψ is defined on the whole of P. (Also we may observe $\psi(P) = \phi(A) = B$ as claimed). It just remains to check that ψ is $\leq$-order-preserving.

Consider $x, y \in P$ with $x \leq y$. Clearly x, y are in the same component of P. If $x = y$, then clearly $\psi(x) = \psi(y)$, and we are done. So assume that $x < y$, and that x, y are in the chain $\mathcal{C}'$. If $\mathcal{C}'$ intersects A, then there are two possibilities - it intersects at the chain C_i or at a single point x_i. (Either way this chain has intervals defined on it).

If $|\mathcal{C}' \cap A| = 1$, then note that $\mathcal{C}'$ is a whole interval, and ψ maps the whole of $\mathcal{C}'$ onto one point $\psi(\mathcal{C}') = \phi(x_i)$, where $\{x_i\} = \mathcal{C}' \cap A$. In this case $\psi(x) = \phi(x_i) = \psi(y)$, so indeed $\psi(x) \leq \psi(y)$.

If $|\mathcal{C}' \cap A| > 1$, say $\mathcal{C}' \cap A = C_i$, then $\mathcal{C}'$ was split into k_i intervals. If x, y are in the same interval I of $\mathcal{C}'$, then $\psi(x) = \phi(a) = \psi(y)$, where $\{a\} = I \cap A$. So indeed $\psi(x) \leq \psi(y)$. Otherwise, $x \in I_x,\ y \in I_y$ with $I_x < I_y$. Say $I_x \cap A = \{a_x\},\ I_y \cap A = \{a_y\}$, then $\psi(x) = \phi(a_x)$ and $\psi(y) = \phi(a_y)$. Since $a_x \leq a_y$ (in fact $a_x < a_y$) and ϕ was $\leq$-order-preserving, we have $\phi(a_x) \leq \phi(a_y)$, so indeed $\psi(x) \leq \psi(y)$.

Otherwise, $\mathcal{C}'$ does not intersect A. So $x, y \in P \setminus (\bigcup I)$, thus $\psi(x) = \phi(a') = \psi(y)$ for the chosen $a' \in A$, so indeed $\psi(x) \leq \psi(y)$.

From now on we may assume that P is connected.

Proposition 21 *Let P be a countable tree (or inverted tree). Then P is* $\mathrm{HH}_{\leq}$.

PROOF. The method is the same as that used for Proposition 10, the corresponding result for trees for the strict-order case.

So, similarly to Proposition 10, let $f : A \to B$ be a $\leq$-homomorphism between finite subsets A, B of P, and let $x \in P \setminus A$, we wish to extend f for x. As before, let $A_{(<x)} := \{z \in A : z < x\}$, $A_{(>x)} := \{z \in A : z > x\}$. Now let $B_{(\leq)} := f(A_{(<x)})$, $B_{(\geq)} := f(A_{(>x)})$. Note that $B_{(\leq)} \leq B_{(\geq)}$ since $A_{(<x)} \leq A_{(>x)}$ (in fact $A_{(<x)} < A_{(>x)}$) and f preserves order.

As before, finding a suitable image for x is a matter of checking the possible cases. Since for all $z \in A,\ z \neq x$, verifying that the extension is $\leq$-order-preserving means checking that if $z < x$, then $f(z) \leq f(x)$; and if $z > x$, then $f(z) \geq f(x)$. In each case this is quite simple.

If $A_{(<x)}$ is nonempty, then since P is a tree, it is a chain, and since f is $\leq$-order-preserving, $B_{(\leq)}$ is also a chain. Since $B_{(\leq)}$ is finite, it has a maximum element, call this y. If $A_{(<x)}$ is empty, but $A_{(>x)}$ is not, then choose y to be a lower bound of $B_{(\geq)}$. If $B_{(\geq)}$ has a minimum element then we can use this, but otherwise any lower bound will do (since P is a tree, certainly lower bounds exist). If $A_{(<x)}, A_{(>x)}$ are both empty, then choose any $y \in P$.

Define $f(x) = y$. In each case the extension is a $\leq$-homomorphism. (But we may note that it is certainly not always one-to-one). Since P is countable, we can extend f in countably many steps to get a $\leq$-homomorphism $\psi : P \to P$ which extends f. So P is $\mathrm{HH}_{\leq}$.

Similarly for inverted trees.

We may now assume that P is not a tree or an inverted tree, so it has some nontrivial antichains with upper bounds and some with lower bounds. Recall the definitions of $\times$-sets and their midpoints introduced when looking at $\mathrm{IH}_<$ posets.

Lemma 22 *If P is an $\mathrm{IH}_\leq$ poset, then either:*

(a) every $\times$-set has a midpoint; or
(b) no $\times$-set has a midpoint.

PROOF. Exactly as for $\mathrm{IH}_<$ posets, Lemma 12.

Proposition 23 *Let P be a countable poset such that all finite subsets have upper and lower bounds, and every $\times$-set has a midpoint. Then P is $\mathrm{HH}_\leq$.*

PROOF. The set up is as in Proposition 21. Let $f : A \to B$ be a $\leq$-homomorphism between finite subsets, and let $x \in P \setminus A$. Define $A_{(<x)}, A_{(>x)}, B_{(\leq)}, B_{(\geq)}$ as before, and note that $B_{(\leq)} \leq B_{(\geq)}$. We wish to find z such that $B_{(\leq)} \leq z \leq B_{(\geq)}$. Then we can define $f(x) = z$, and this is clearly $\leq$-order-preserving. Then as in previous arguments, since P is countable, we can extend f in countably many steps to get a $\leq$-homomorphism $\psi : P \to P$ which extends f. So P is $\mathrm{HH}_\leq$.

The main part of the proof lies in showing that for any finite $X, Y \subset P$ with $X \leq Y$, there exists z such that $X \leq z \leq Y$. We use the same method as that used in the second part of Proposition 13, i.e. induction on the size of $|X| + |Y|$. The proof is almost identical, so we will just describe the steps which differ.

Consider the proof of the claim that we can assume X, Y are antichains and $|X|, |Y| \geq 2$. We first suppose $X = \emptyset$ (but $Y \neq \emptyset$). Here a suitable z is given by any lower bound of Y, which we know to exist since every finite subset of P has a lower bound. (In particular, if Y has a minimum element, then we can take this as the lower bound). Next if $X = \{x\}$, then $X \leq x \leq Y$, so simply choose $z = x$. Similarly, $|Y| \geq 2$. The last part of the proof of the claim is identical.

In the final part of the proof we must take some care since our induction hypothesis now deals with the non-strict order rather than the strict order. However essentially the proof is the same, just with some orders weakened.

Proposition 24 *Let P be a countable poset such that all finite subsets have upper and lower bounds, and no $\times$-set has a midpoint. Then P is* $\mathrm{HH}_{\leq}$.

PROOF. The proof is similar to Proposition 14, the corresponding result for the strict-order case, for posets such that no $\times$-sets have midpoints. Assume that $\times$-sets do exist, otherwise vacuously all $\times$-sets have midpoints so we may use Proposition 23.

The set up is as in Proposition 21. Let $f : A \to B$ be a $\leq$-homomorphism between finite subsets, and let $x \in P \setminus A$, we wish to extend f for x. Define $A_{(<x)}, A_{(>x)}, B_{(\leq)}, B_{(\geq)}$ as before, and note that $B_{(\leq)} \leq B_{(\geq)}$. As before, finding a suitable image for x is a matter of checking the possible cases. Recall from Proposition 21 that verifying that the extension is $\leq$-order-preserving means checking that if $z < x$, then $f(z) \leq f(x)$; and if $z > x$, then $f(z) \geq f(x)$. In each case this is quite simple.

Firstly, suppose $B_{(\leq)}, B_{(\geq)}$ are both nonempty. As in Proposition 14, either $A_{(<x)}$ or $A_{(>x)}$ must be a chain, else we can find a $\times$-set with midpoint x, which gives a contradiction. So assume that $A_{(<x)}$ is a chain, then so is $B_{(\leq)}$ since f is $\leq$-order-preserving. Let y be the maximum element of the finite chain $B_{(\leq)}$.

Now suppose that exactly one of $B_{(\leq)}, B_{(\geq)}$ is empty, without loss of generality say $B_{(\leq)} = \emptyset$. Choose y to be a lower bound of $B_{(\geq)}$. Finally, if $B_{(\leq)}, B_{(\geq)}$ are both empty, then choose any $y \in P$.

Define $f(x) = y$, in each case the extension is a $\leq$-homomorphism. Since P is countable, we can extend f in countably many steps to get a $\leq$-homomorphism $\psi : P \to P$ which extends f. So P is $\mathrm{HH}_{\leq}$.

Proposition 25 *A countable (finite or infinite) poset P is* $\mathrm{IH}_{\leq}$ *if and only if it is one of the following:*

(i) *A disjoint union of $n \in \mathbb{N}^*$ incomparable countable chains (possibly of different lengths, including trivial chains);*
(ii) *A tree (or inverted tree);*
(iii) *A poset such that all finite subsets have upper and lower bounds, and either*
  (a) *every $\times$-set has a midpoint; or*
  (b) *no $\times$-set has a midpoint.*

PROOF. ($\Rightarrow$): By Proposition 19(a), if P is disconnected then it is a disjoint union of incomparable countable chains (possibly of different lengths) and isolated points, so (i) holds.

Otherwise, assume P is connected, and is not a chain. Then there exists a 2-element antichain, with either an upper or lower bound, say it has a lower bound. Now by Proposition 19(b), any finite subset of P has a lower bound.

By Proposition 19(b), either every 2-element antichain of P has an upper bound, or none does. If no 2-element antichain of P has an upper bound, then P is a tree, so (ii) holds.

If we instead had that every finite subset has an upper bound, but no antichain has a lower bound, then P is an inverted tree, so again (ii) holds.

Otherwise, every 2-element antichain of P has upper and lower bounds, so every finite subset of P has upper and lower bounds. So the first part of (iii) holds. By Lemma 22, if P is $\mathrm{IH}_{\leq}$, then indeed either (a) or (b) holds.

The converse follows from Propositions 20, 21, 23, and 24, and the fact that if P is $\mathrm{HH}_{\leq}$, then it is $\mathrm{IH}_{\leq}$.

Corollary 26 *The classes of countable posets* $\mathrm{IH}_{\leq}$*,* $\mathrm{MH}_{\leq}$*,* $\mathrm{HH}_{\leq}$ *are all equal.*

Thus for countable posets we have the picture:

$$\begin{array}{c} \mathrm{IH}_{\leq} = \mathrm{MH}_{\leq} = \mathrm{HH}_{\leq} \\ | \\ \mathrm{IM} = \mathrm{MM} \\ | \\ \mathrm{II} \end{array}$$

Here $\mathrm{II} := \mathrm{II}_{<} = \mathrm{II}_{\leq}$, $\mathrm{IM} := \mathrm{IM}_{<} = \mathrm{IM}_{\leq}$, $\mathrm{MM} := \mathrm{MM}_{<} = \mathrm{MM}_{\leq}$ by Lemma 18.

Combining the classes with $\leq$-order-preserving maps and those with $<$-order-preserving maps, for posets we have the overall picture:

$$\begin{array}{c} \mathrm{IH}_{\leq} = \mathrm{MH}_{\leq} = \mathrm{HH}_{\leq} \\ | \\ \mathrm{IH}_{<} = \mathrm{MH}_{<} = \mathrm{HH}_{<} = \mathrm{IM} = \mathrm{MM} \\ | \\ \mathrm{II} \end{array}$$

Note that if we specify to just look at countable trees, then these pictures remain the same.

The picture is now interesting even if we just look at finite structures. The only finite IH posets are the finite antichains, which are also II. However there are many finite $\mathrm{IH}_{\leq}$ posets which are not antichains. So for finite posets (and for finite trees) we have the picture:

$$
\begin{array}{c}
\mathrm{IH}_{\le} = \mathrm{MH}_{\le} = \mathrm{HH}_{\le} \\
| \\
\mathrm{IH}_{<} = \mathrm{MH}_{<} = \mathrm{HH}_{<} = \mathrm{IM} = \mathrm{MM} = \mathrm{II}
\end{array}
$$

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