Random preorders

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Abstract

A random preorder on n elements consists of linearly ordered equivalence classes called *blocks*. We investigate the block structure of a preorder chosen uniformly at random from all preorders on n elements as $n \to \infty$.

1 Introduction

Let R be a binary relation on a set X. We say R is *reflexive* if $(x,x) \in R$ for all $x \in X$. We say R is *transitive* if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$. A *partial preorder* is a relation R on X which is reflexive and transitive. A relation R is said to satisfy *trichotomy* if, for any $x,y \in X$, one of the cases $(x,y) \in R$, $x = y$, or $(y,x) \in R$ holds. We say that R is a *preorder* if it is a partial preorder that satisfies trichotomy. The members of X are said to be the *elements* of the preorder.

A relation R is *antisymmetric* if, whenever $(x,y) \in R$ and $(y,x) \in R$ both hold, then $x = y$. A relation R on X is a *partial order* if it is reflexive, transitive, and antisymmetric. A relation is a *total order*, if it is a partial order which satisfies trichotomy. Given a partial preorder R on X, define a new relation S on X by the rule that $(x,y) \in S$ if and only if both (x,y) and (y,x) belong to R. Then S is an equivalence relation. Moreover, R induces a partial order $\bar{x}$ on the set of equivalence classes of S in a natural way: if $(x,y) \in R$, then $(\bar{x},\bar{y}) \in \overline{R}$, where $\bar{x}$ is the S-equivalence class containing x and similarly for y. We will call an S-equivalence class a *block*. If R is a preorder, then the relation $\overline{R}$ on the equivalence classes of S is a total order. See Section 3.8 and question 19 of Section 3.13 in [4] for more on the above definitions and results.

Preorders are used in [6] to model the voting preferences of voters. (A different but equivalent definition of preorders is used in [6], where they are called weak orders.) We suppose that there are n candidates and m voters. Suppose that X is a finite set representing a collection of candidates. Let R_i, $i = 1, 2, \dots, m$, be a set of weak orders on X. Then $(x, y) \in R_i$ means that the ith voter prefers candidate y to candidate x. The R_i blocks correspond to sets of candidates to which voter i is indifferent.

Let $p(n)$ denote the number of preorders possible on a set of n elements. The assumption is made in [6] that each voter chooses his voting preference uniformly at random from all of the $p(n)$ possibilities independently of the other voters. An algorithm for generating a random preorder is given in [6] and the ideas behind the algorithm are used to derive a formula for the probability of the occurrence of "Condorcet's paradox". See [5] for a survey of assumptions on voter preferences used in the study of Condorcet's paradox.

We are interested in the size of the blocks in a random preorder. Let B_1 be the size of the first block, let B_2 be the size of the second, and let B_i be the size of the ith block. If the preorder has N blocks we define $B_i = 0$ for $i > N$. It is an identity that

$$\sum_{i=1}^{\infty} B_i = n. \tag{1.1}$$

We can represent a preorder on the set X by the sequence $(B_1, B_2, \dots)$, where the B_i are disjoint and $\bigcup_i B_i = X$. A related combinatorial object to preorders is set partitions, for which the blocks are not ordered. The block structure of random set partitions has been studied in [8].

In this paper we look at the block structure of a random preorder. We give asymptotic estimates of the number of blocks, the size of a typical block, and the number of blocks of a particular size. We are able to show that the maximal size of a block asymptotically takes on one of two values.

Let $S(n, k)$ denote the Stirling number of the second kind. Note that the number of preorders with exactly r blocks is $p(n, r) = r!S(n, r)$ and therefore $p(n) = \sum_{r=1}^{n} r!S(n, r)$.

The identity

$$\sum_{n=0}^{\infty} \frac{S(n, r) z^n}{n!} = \frac{(e^z - 1)^r}{r!} \tag{1.2}$$

is proven in Proposition (5.4.1) of [4] using inclusion-exclusion. The following consequence of (1.2) will be useful. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series, then we use the notation $[z^n]\, f(z)$ to denote a_n.

Lemma 1 *For any sequence θ_r, $1 \le r \le n$,*

$$\sum_{r=1}^{n} \theta_r p(n,r) = n!\,[z^n]\left(\sum_{n=0}^{\infty} \theta_n (e^z-1)^n\right).$$

Proof We observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{r=1}^{n} \theta_r p(n,r)\right)\frac{z^n}{n!} &= \sum_{r=1}^{\infty}\left(\sum_{n=r}^{\infty}\frac{p(n,r)z^n}{n!}\right)\theta_r \\
&= \sum_{r=1}^{\infty}\left(\sum_{n=r}^{\infty}\frac{S(n,r)z^n}{n!}\right) r!\,\theta_r \\
&= \sum_{r=1}^{\infty}\theta_r (e^z-1)^r.
\end{aligned}$$

The lemma follows immediately. ∎

If we take $\theta_r = 1$ in Lemma 1, then we find that

$$p(n) = n![z^n]\,(2-e^z)^{-1},$$

an identity proved in [2]. The singularity of smallest modulus of $(2-e^z)^{-1}$ occurs at $z = \log 2$ with residue

$$\lim_{z\to\log 2}\left(\frac{z-\log 2}{2-e^z}\right) = \lim_{z\to\log 2}\left(\frac{1}{-e^z}\right) = -\frac{1}{2}, \tag{1.3}$$

by l'Hôpital's rule. So the function

$$\frac{1}{2-e^z} + \frac{1}{2(z-\log 2)}$$

is analytic in a circle with centre at the origin and the next singularities of $(2-e^z)^{-1}$ (at $\log 2 \pm 2\pi i$) on the boundary. Thus

$$p(n) \sim \frac{n!}{2}\left(\frac{1}{\log 2}\right)^{n+1}, \tag{1.4}$$

and indeed it follows from Theorem 10.2 of [7] that the difference between the two sides is $o((r-\varepsilon)^{-n})$, where $r = |\log 2 + 2\pi i|$; that is, exponentially small. An exact expression for $(2-e^z)^{-1}$ is given in [1] in terms of its singularities and the truncation error from using only a finite number of singularities is estimated.

2 The number of blocks

We denote the number of blocks of a random preorder on n elements by X_n. In terms of the block sizes B_i we may express X_n as $X_n = \sum_{i=1}^{\infty} I[B_i > 0]$, where $I[B_i > 0]$ is the indicator variable that the ith block has positive size. In this section we give asymptotics for X_n.

The kth falling factorial of a real number x is defined to be $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ and the kth falling moment of X_n to be

$$\mathbb{E}(X_n)_k = \mathbb{E}X_n(X_n-1)(X_n-2)\cdots(X_n-k+1). \tag{2.5}$$

Define λ_n to be

$$\lambda_n = \frac{n}{2\log 2}. \tag{2.6}$$

We will show that $\mathbb{E}(X_n)_k \sim \lambda_n^k$ for each fixed $k \geq 0$, where we use the notation $a_n \sim b_n$ for sequences a_n, b_n to mean $\lim_{n\to\infty} a_n/b_n = 1$. By a standard argument using Chebyshev's inequality, the asymptotics of the first two moments implies that $X_n \overset{\text{a.a.s.}}{\sim} \frac{n}{2\log 2}$, where we write $X_n \overset{\text{a.a.s.}}{\sim} a_n$ (X_n converges to a_n asymptotically almost surely) to mean $\lim_{n\to\infty} \mathbb{P}(|X_n/a_n - 1| > \varepsilon) = 0$ for all $\varepsilon > 0$.

Theorem 1 *The kth falling moment of the number of blocks of a random preorder equals*

$$\mathbb{E}(X_n)_k = \frac{k!n!}{p(n)}[z^n]\frac{(e^z-1)^k}{(2-e^z)^{k+1}}. \tag{2.7}$$

It follows that for fixed k

$$\mathbb{E}(X_n)_k \sim \lambda_n^k \tag{2.8}$$

and that

$$X_n \overset{\text{a.a.s.}}{\sim} \lambda_n,$$

where λ_n is defined by (2.6).

Proof In order to prove (2.7) it suffices to note that

$$\mathbb{E}(X_n)_k = \sum_{r=1}^{n} \frac{p(n,r)}{p(n)}(r)_k = \frac{1}{p(n)}\sum_{r=1}^{n} p(n,r)(r)_k,$$

to apply Lemma 1 with $\theta_r = (r)_k$, and to observe that

$$\sum_{n=0}^{\infty} (n)_k x^n = \frac{k!x^k}{(1-x)^{k+1}}.$$

We now proceed to show (2.8). An analysis similar to (1.3) shows that

$$\lim_{z\to\log 2}\frac{(z-\log 2)^{k+1}(e^z-1)^k}{(2-e^z)^{k+1}}=\left(-\frac{1}{2}\right)^{k+1}$$

and that

$$\frac{(e^z-1)^k}{(2-e^z)^{k+1}}-\frac{(-1/2)^{k+1}}{(z-\log 2)^{k+1}}$$

is analytic on any disc of radius less than $|\log 2+2\pi i|$. Singularity analysis (Section 11 of [7]) implies that

$$\begin{aligned}[z^n]\frac{(e^z-1)^k}{(2-e^z)^{k+1}} &\sim \left(\frac{1}{2\log 2}\right)^{k+1}[z^n](1-z/\log 2)^{-k-1}\\ &\sim \left(\frac{1}{2\log 2}\right)^{k+1}\frac{n^k}{\Gamma(k+1)(\log 2)^n},\end{aligned}\tag{2.9}$$

where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$. Therefore,

$$\mathbb{E}(X_n)_k\sim\frac{n!}{p(n)}\left(\frac{1}{2\log 2}\right)^{k+1}\frac{n^k}{(\log 2)^n}\sim\lambda_n^k,$$

where we have used (1.4) and $\Gamma(k+1)=k!$.

The variance of X_n is asymptotically $\mathrm{Var}(X_n)=\mathbb{E}X_n(X_n-1)+\mathbb{E}X_n-(\mathbb{E}X_n)^2=\lambda_n^2+o(\lambda_n^2)+(\lambda_n+o(\lambda_n))-(\lambda_n+o(\lambda_n))^2=\lambda_n(1+o(\lambda_n))$. The conclusion that $X_n\overset{\text{a.a.s.}}{\sim}\lambda_n$ is a consequence of

$$\mathbb{P}(|X_n/\lambda_n-1|>\varepsilon)=\mathbb{P}(|X_n-\lambda_n|>\varepsilon\lambda_n)\leq\mathrm{Var}(X_n)/(\varepsilon\lambda_n)^2=o(1).$$

■

As a random variable Z with Poisson(λ_n) distribution has falling moments exactly equal to $\mathbb{E}(Z)_k=\lambda_n^k$, and $(Z-\lambda_n)/\sqrt{\lambda_n}$ converges weakly to the standard normal distribution if $\lambda_n\to\infty$, (2.8) indicates that $(X_n-\lambda_n)/\sqrt{\lambda_n}$ should have a distribution that is approximately normal. Asymptotic normality could be a subject for future research.

3 The size of a typical block

Because the blocks in a random preorder are linearly ordered, we may take B_1 as the size of a typical block. Given a preorder $(B_1, B_2, \ldots)$ on X, we may define a new preorder on $X \backslash B_1$ by the sequence $(B_2, B_3, \ldots)$. This operation can be reversed: given a preorder on $X \backslash B_1$, $(B_2, B_3, \ldots)$, we can insert B_1 to get the original preorder on X. The above correspondence implies

$$\mathbb{P}(B_1 = k) = \binom{n}{k} \frac{p(n-k)}{p(n)}$$

and for fixed k the asymptotic (1.4) gives

$$\mathbb{P}(B_1 = k) \sim \binom{n}{k} \frac{(n-k)!}{n!} (\log 2)^k = \frac{(\log 2)^k}{k!}. \tag{3.10}$$

It is easily checked that the distribution defined by the right hand side of (3.10) is the same as the distribution of the conditioned random variable $(Z|Z > 0)$, where Z is Poisson$(\log 2)$ distributed.

We will use an argument similar to the one above and the results of Section 2 to show that the distribution of fixed block sizes are asymptotically i.i.d. and distributed as $(Z|Z > 0)$.

Theorem 2 *Let a finite set of indices $i_1, i_2, \ldots, i_L$ and a sequence of nonnegative integers $a_1, a_2, \ldots, a_L$ be given. Then*

$$\mathbb{P}(B_{i_1} = a_1, B_{i_2} = a_2, \ldots, B_{i_L} = a_L) \sim \prod_{i=1}^{L} \frac{(\log 2)^{a_i}}{a_i!}.$$

That is, the distribution of the B_{i_j} converges weakly to an i.i.d. sequence of random variables distributed as $(Z|Z > 0)$, where Z is Poisson$(\log 2)$ distributed.

Proof Given a preorder with blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_L}$ on X, we can form a new preorder by $(B_1, \ldots, B_{i_1 - 1}, B_{i-1+1}, \ldots, B_{i_2 - 1}, B_{i_2 + 1} \cdots)$ on $X \backslash \bigcup_{l=1}^{L} B_{i_l}$. On the other hand, a preorder $(B_1, \ldots, B_{i_1 - 1}, B_{i-1+1}, \ldots, B_{i_2 - 1}, B_{i_2 + 1} \cdots)$ on $X \backslash \bigcup_{l=1}^{L} B_{i_l}$ forms a valid preorder $(B_1, B_2, \ldots)$ on X by the insertion of the blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_L}$ if and only if $(B_1, \ldots, B_{i_1 - 1}, B_{i-1+1}, \ldots, B_{i_2 - 1}, B_{i_2 + 1} \ldots)$ is a preorder with at least

$i_L - L$ nonempty blocks. Therefore, with b defined as $b = \sum_{l=1}^{L} a_l$,

$$\begin{aligned} & \mathbb{P}(B_{i_1} = a_1, B_{i_2} = a_2, \ldots, B_{i_L} = a_L) \\ = & \binom{n}{a_1, a_2, \ldots, a_L, n-b} \frac{\sum_{r=i_L-L}^{\infty} p(n-b,r)}{p(n)} \\ = & \binom{n}{a_1, a_2, \ldots, a_L, n-b} \frac{p(n-b)}{p(n)} \mathbb{P}(X_{n-b} \geq a_L - L). \end{aligned} \tag{3.11}$$

The probability in (3.11) approaches 1 because of Theorem 1. The other factors have asymptotics that give the theorem. ■

4 The number of blocks of fixed size

Define $X_n^{(s)}$ to be the number of blocks of size $s = s(n)$ in a random preorder on n elements. Define $\lambda_n^{(s)}$ to be

$$\lambda_n^{(s)} = \frac{(\log 2)^{s-1} n}{2s!}. \tag{4.12}$$

Theorem 3 *The kth falling moment of the number of s-blocks of a random preorder equals*

$$\mathbb{E}(X_n^{(s)})_k = \frac{k!n!}{p(n)(s!)^k} [z^{n-ks}] \sum_{j=0}^{k} \frac{(k)_j (e^z - 1)^{k-j}}{j!(2-e^z)^{k-j+1}}. \tag{4.13}$$

It follows that for fixed k and $s = o(n)$ such that $\lambda_n^{(s)} \to \infty$,

$$\mathbb{E}(X_n^{(s)})_k \sim (\lambda_n^{(s)})^k$$

and that

$$X_n^{(s)} \overset{\text{a.a.s.}}{\sim} \lambda_n^{(s)}, \tag{4.14}$$

where $\lambda_n^{(s)}$ is defined by (4.12).

Proof Let $p_s(n,k)$ be the number of preorders on n elements with exactly k blocks of size s. The kth falling moment of $X_n^{(s)}$ is

$$\mathbb{E}(X_n^{(s)})_k = \frac{1}{p(n)} \sum_{r=0}^{\infty} (r)_k p_s(n,r).$$

The quantity $\sum_{r=0}^{\infty}(r)_k p_s(n,r)$ counts the number of preorders with k labelled s-blocks, where each of the labelled s-blocks is given a unique label from the set $\{1,2,\ldots,k\}$. This number is also counted by: first, choosing k s-blocks to be the ones marked; second, forming a preorder on the $n-ks$ remaining elements with r blocks; third, inserting the s-blocks into the preorder in the order they were chosen in one of $\binom{k+r}{k}$ ways; fourth, marking the inserted s-blocks in one of $k!$ ways. We therefore have

$$\begin{aligned}\mathbb{E}(X_n^{(s)})_k &= \frac{1}{p(n)}\sum_{r=1}^{\infty}\binom{n}{s}\binom{n-s}{s}\cdots\binom{n-(k-1)s}{s}p(n-ks,r)\binom{k+r}{k}k! \\ &= \frac{n!}{p(n)(s!)^k(n-ks)!}\sum_{r=1}^{\infty}p(n-ks,r)(k+r)_k \\ &= \frac{n!}{p(n)(s!)^k}[z^{n-ks}]\sum_{n=0}^{\infty}(k+n)_k(e^z-1)^n\end{aligned} \tag{4.15}$$

where we have made use of Lemma 1 at (4.15). We use the identity

$$\sum_{n=0}^{\infty}(k+n)_k x^n = \frac{d^k}{dx^k}\frac{x^k}{1-x} \quad = \quad \sum_{j=0}^{k}\binom{k}{j}\frac{(k-j)!(k)_j x^{k-j}}{(1-x)^{k-j+1}}$$

in (4.15), which follows from the formula $\frac{d^k}{dx^k}uv = \sum_{j=0}^{k}\binom{k}{j}\frac{d^j}{dx^j}u\frac{d^{k-j}}{dx^{k-j}}v$ for functions $u(x)$ and $v(x)$. After substitution of the identity and simplification (4.15) becomes (4.13).

In (4.13), the singularity of largest degree occurs at $z=\log 2$ when $j=0$. The asymptotics of $\mathbb{E}(X_n^{(s)})_k$ are given by

$$\begin{aligned}\mathbb{E}(X_n^{(s)})_k &\sim \frac{n!k!}{p(n)(s!)^k}[z^{n-ks}]\frac{(e^z-1)^k}{(2-e^z)^{k+1}} \\ &\sim \frac{2k!(\log 2)^{n+1}}{(s!)^k}\left(\frac{1}{2\log 2}\right)^{k+1}\frac{(n-ks)^k}{\Gamma(k+1)(\log 2)^{n-ks}} \\ &\sim \left(\frac{(\log 2)^{s-1}n}{2s!}\right)^k,\end{aligned}$$

where we have used (1.4), (2.9), and the assumption $s=o(n)$. The almost sure convergence result (4.14) is an application of Chebyshev's inequality as in the proof of Theorem 1. ■

One would expect from Theorem 3 that the distribution of $\frac{X_n^{(s)}-\lambda_n(s)}{\sqrt{\lambda_n(s)}}$ converges weakly to a standard normal distribution as long as $\lambda_n(s)\to\infty$, where $\lambda_n(s)=\frac{(\log 2)^{s-1}n}{2s!}$. This could be the subject of further investigations.

The method of the proof of Theorem 3 can be used to derive asymptotics of joint falling moments. For example, $\mathbb{E}((X_n^{(s_1)})_{k_1}(X_n^{(s_2)})_{k_2})\sim(\lambda_n^{(s_1)})^{k_1}(\lambda_n^{(s_2)})^{k_2}$ for fixed s_1, s_2, k_1, k_2.

Observe that $\sum_{s=1}^{\infty}s\lambda_n^{(s)}=n$ and $\sum_{s=1}^{\infty}\lambda_n^{(s)}=\lambda_n$, showing that Theorem 4.13 agrees with (1.1) and Theorem 1, respectively, and indicating that Theorem 3 gives a good picture of the block structure of a random preorder.

5 Maximal block size

We are also able to estimate closely the the maximum size of a block in a random preorder. Define μ_n to be

$$\mu_n=\max\left\{s:\lambda_n^{(s)}\geq 1\right\}$$

and define

$$\nu_n=\begin{cases}\mu_n & \text{if } \lambda_n(\mu_n)\geq\sqrt{\mu_n},\\ \mu_n-1 & \text{if } \lambda_n(\mu_n)<\sqrt{\mu_n}.\end{cases}\tag{5.16}$$

Theorem 4 *Let $M_n=\max_{i\geq 1}B_i$ be the maximal size of a block in a random preorder. Let ν_n be defined by (5.16). Then $\nu_n\sim\log n/\log\log n$ and*

$$\mathbb{P}(M_n\in\{\nu_n,\nu_n+1\})\to 1\ \text{ as } n\to\infty.\tag{5.17}$$

Proof Clearly, $\lambda_n^{(s)}$ is monotone decreasing in s for $s\geq 2$. Taking the logarithm of $\lambda_n^{(s)}$ produces

$$\begin{aligned}\log\lambda_n^{(s)} &= \log n+(s-1)\log\log 2-\log s!-\log 2\\ &= \log n-s\log s+O(s).\end{aligned}\tag{5.18}$$

Plugging $s=\frac{\log n}{\log\log n}$ into (5.18) gives

$$\log\lambda_n\left(\frac{\log n}{\log\log n}\right)=\frac{\log n\log\log\log n}{\log\log n}+O\left(\frac{\log n}{\log\log n}\right)\to\infty,$$

from which it follows that for large enough n, $\mu_n > \log n/\log\log n$. On the other hand, if we plug $\frac{\log n}{\log\log n}\left(1+\frac{2\log\log\log n}{\log\log n}\right)$ into the right hand side of (5.18) we get

$$\begin{aligned}
& \log\lambda_n\left(\frac{\log n}{\log\log n}\left(1+\frac{2\log\log\log n}{\log\log n}\right)\right) \\
= \ & \log n - \frac{\log n}{\log\log n}\left(1+\frac{2\log\log\log n}{\log\log n}\right)\left(\log\log n - \log\log\log n + O\left(\frac{\log\log\log n}{\log\log n}\right)\right) \\
& + O\left(\frac{\log n}{\log\log n}\right) \\
= \ & -\frac{\log n\log\log\log n}{\log\log n} + O\left(\frac{\log n(\log\log\log n)^2}{(\log\log n)^2}\right) \to -\infty,
\end{aligned}$$

so that $\mu_n < \frac{\log n}{\log\log n}\left(1+\frac{2\log\log\log n}{\log\log n}\right)$ for large enough n. We have shown that

$$\frac{\log n}{\log\log n} < \mu_n < \frac{\log n}{\log\log n}\left(1+\frac{2\log\log\log n}{\log\log n}\right)$$

for large enough n and, in particular, that $\mu_n \sim \frac{\log n}{\log\log n}$ and $\nu_n \sim \frac{\log n}{\log\log n}$.

Define the index sets

$$\mathcal{N}_1 = \{n \geq 1 : \lambda_n(\mu_n) \geq \sqrt{\mu_n}\}$$

and

$$\mathcal{N}_2 = \{n \geq 1 : \lambda_n(\mu_n) < \sqrt{\mu_n}\}.$$

We prove (5.17) first for indices going to infinity in $\mathcal{N}_1$ and then for indices going to infinity in $\mathcal{N}_2$.

When $n \to \infty$ in $\mathcal{N}_1$, $\lambda_n(\nu_n) \geq \sqrt{\mu_n} \to \infty$ as $n \to \infty$, so the proof of Theorem 3 gives $\mathbb{P}(X_n^{(\nu_n)} > 0) \to 1$ and so $\mathbb{P}(M_n < \nu_n) \to 0$. The ratio $\lambda_n^{(\mu_n+1)}/\lambda_n^{(\mu_n+2)} = (\mu_n+2)/\log 2 \to \infty$ and $\lambda_n^{(\mu_n+1)} < 1$ imply $\lambda_n^{(\mu_n+2)} \to 0$. We will show, furthermore, that

$$\sum_{s\geq\mu_n+2} \mathbb{E}(X_n^{(s)}) = o(1), \tag{5.19}$$

which implies $\mathbb{P}(M_n > \nu_n + 1) \to 0$. By Theorem 3, after some simplification, for all $s \in [1,n]$

$$\mathbb{E}(X_n^{(s)}) \quad = \quad \frac{n!}{p(n)s!}[z^{n-s}](2-e^z)^{-2}$$

$$\begin{aligned} &\le \frac{Kn!}{p(n)s!}\frac{n}{(\log 2)^{n-s}} \qquad (5.20) \\ &\le K'\frac{(\log 2)^{s-1}n}{s!} \end{aligned}$$

for constants $K, K' > 0$, where we have used the $O(\cdot)$ version of singularity analysis [7] at (5.20). The ratios $\frac{n(\log 2)^{s+1}/(s+1)!}{n(\log 2)^s/(s)!} = \frac{\log 2}{s+1}$ are less than some fixed $\rho < 1$ for all $s \ge \mu_n + 2$ for large enough n, so that for n large enough,

$$\sum_{s \ge \mu_n+2} \mathbb{E}(X_n^{(s)}) \le K' \sum_{s \ge \mu_n+2} \frac{n(\log 2)^{s-1}}{s!} \le \frac{K'\lambda_n^{(\mu_n+2)}}{1-\rho} \to 0. \qquad (5.21)$$

When $n \to \infty$ in $\mathcal{N}_2$, $\lambda_n^{(\mu_n)} \ge 1$ and $\lambda_n^{(\mu_n-1)}/\lambda_n^{(\mu_n)} = \mu_n/\log 2 \to \infty$ give $\lambda^{(\nu_n)} \to \infty$, hence $\mathbb{P}(M_n < \nu_n) \to 0$. On the other hand, $\lambda^{(\mu_n)} < \sqrt{\mu_n}$ and $\lambda_n^{(\mu_n)}/\lambda_n^{(\mu_n+1)} = (\mu_n+1)/\log 2 \to \infty$ give $\lambda^{(\mu_n+1)} = O(\mu_n{}^{-1/2}) = o(1)$ and an argument like the one showing (5.21) results in $\mathbb{P}(M_n > \nu_n + 1) \to 0$. ■

Asymptotic two-point concentration theorems are well known from random graph theory. See Theorem 7, page 260 of [3] for such a result regarding clique number.

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