Altitude and chromatic number

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Abstract

Mynhardt *et al.* defined a graph parameter, which they called *altitude*, to be the largest guaranteed monotonic path in any edge-ordering. This is related to the chromatic index. The analogous vertex parameter is equal to the chromatic number. However, if we use circulat instead of linear orderings, we obtain something new.

Define a parameter $a(G)$ of a graph G as follows: it is the largest integer k such that, for any ordering of the vertices of G, there is a monotone path with k vertices.

If we replace vertices by edges in the definition, we obtain the parameter called *altitude* by Mynhardt *et al.* [1]. The altitude $\alpha(G)$ of a graph G is closely connected with its edge-chromatic number. It turns out that the parameter $a(G)$ is even more closely connected to the vertex-chromatic number, by the following (folklore?) result:

Proposition 1 $a(G) = \chi(G)$.

Proof Suppose that we have a vertex colouring of G with k colours. Order the colours, then order the vertices within each colour class. Clearly a monotone path has all colours distinct, so has at most k vertices. Thus $a(G) \leq \chi(G)$.

Conversely, suppose we have an ordering of the vertices so that the longest monotone path has l vertices. For each vertex v, define $c(v)$ to be the largest number of vertices on a monotone path ending at v. Clearly $c(v) \in \{1, \ldots, l\}$. Suppose that v and w are adjacent vertices with $c(v) = c(w)$; without loss, suppose that $v < w$. Then appending w to a monotone path ending at v gives a longer monotone path ending at w, contradicting $c(v) = c(w)$. So c is a proper vertex-colouring. Thus $\chi(G) \leq a(G)$.

The proposition follows. $\square$.

Of course this doesn't trivialise the theory of altitude. An ordering of the edges of G is an ordering of the vertices of $L(G)$; but not every path in $L(G)$ comes from a path in G. So we have

$$\alpha(G) \leq a(L(G)) = \chi(L(G)) = \chi'(G),$$

but examples in [1] show that the first inequality can be strict.

Another possible direction would be to consider the parameter $a^\circ(G)$ defined as follows: it is the largest integer k such that, for any circular ordering of the vertices of G, there is a monotone cycle with k vertices. (Here the vertices are labelled with the elements of a circularly ordered set such as $\mathbb{Z}/(n)$; a monotone cycle is one where the indices traverse the circle once in the forward or backward direction as we walk round the cycle. (An edge of the graph is counted as a cycle of length 2.) The same argument as before shows that $a^\circ(G) \leq \chi(G)$. This time the inequality can be strict: if G is an odd cycle of length at least 5, then $\chi(G) = 3$, but $a^\circ(G)) < 3$ since there are no 3-cycles, so that $a^\circ(G) = 2$. Note that

- $a^\circ(G) \geq \omega(G)$, since the vertices of a clique can be traversed in the induced cyclic order;

- either $a^\circ(G) = 2$, or $a^\circ(G)$ is at least equal to the girth of G.

References

[1] K. Mynhardt, A. Burger, T. Clark, B. Falvai and N. Henderson, Edge orderings, edge colourings and altitude of cubic graphs, talk at 17th Cumberland Conference, Murfreesboro, May 2004.