A family of balanced incomplete-block designs with repeated blocks on which general linear groups act

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In memory of Jack van Lint

Abstract

We give two constructions of a balanced incomplete-block design discovered by van Lint: the design has parameters $(13, 39, 15, 5, 5)$, and has repeated blocks and an automorphism group of order 240. One of these methods can be generalised to produce a large class of designs with the properties of the title.

1 Introduction

In [5], van Lint gave a cyclic construction for a balanced incomplete-block design for 13 treatments in 39 blocks of size 5, of which 24 form pairs of identical blocks. By using `nauty` [6], D. A. Preece [7] discovered that the automorphim group of this design induces a group of order 240 on the treatments. Suspecting that this might be a subgroup of a quotient of $\mathrm{GL}(2, 5)$, we tried to find another direct construction of the design which would make it obvious that it has this large group of automorphisms. We do this, in two different ways, in Section 2.

One of these constructions can be further generalised: for each prime-power q, each integer $m \geq 2$, and each divisor z of $q - 1$, we construct a balanced incomplete-block design on $1 + (q^m - 1)/z$ treatments, having repeated blocks, and having an automorphism group $\mathrm{GL}(m, q)/Z$, where Z is the subgroup of order z of the group of non-zero scalars in $\mathrm{GL}(m, q)$. The actual construction is even more general.

2 The design for 13 treatments

2.1 Construction from an affine plane

The affine plane over GF(5) is shown in Figure 1, using the convention that if P is a point then $\bar{P}$ is the point $-P$. The point $(0,0)$ is called O. The 30 lines of this plane, of which 6 pass through the point O, form the blocks of a balanced square lattice design.

	0	1	2	3	4
0	O	A	B	$\bar{B}$	$\bar{A}$
1	C	E	G	I	K
2	D	J	F	L	H
3	$\bar{D}$	$\bar{H}$	$\bar{L}$	$\bar{F}$	$\bar{J}$
4	$\bar{C}$	$\bar{K}$	$\bar{I}$	$\bar{G}$	$\bar{E}$

Figure 1: The affine plane over GF(5)

Define points P and Q to be equivalent if $Q = \pm P$. The 13 equivalence classes of points will form the treatments in the new design. Write $[P]$ for the equivalence class containing P.

For each line $\mathcal{L}$ of the plane, put $\phi(\mathcal{L}) = \{[P] : P \in \mathcal{L}\}$ and $-\mathcal{L} = \{-P : P \in \mathcal{L}\}$. If $\mathcal{L}$ does not contain O then $\phi(\mathcal{L})$ contains 5 treatments and $\phi(\mathcal{L}) = \phi(-\mathcal{L})$. The 24 sets $\phi(\mathcal{L})$, for lines not containing O, are blocks of the new design. They are shown on the left of Table 1, omitting the [ ]. If $P \neq O$ then P occurs in 5 lines which do not contain O, so $[P]$ occurs 10 times in these blocks. If Q is a multiple of P other than O, P and $-P$ then $[P]$ and $[Q]$ never concur in these blocks, because the line PQ contains O. If Q is any other non-zero point then $[P]$ and $[Q]$ concur in the four blocks $\phi(\mathcal{L})$ for $\mathcal{L} = PQ$, $P\bar{Q}$, $\bar{P}Q$ and $\bar{P}\bar{Q}$.

The remaining 15 blocks have the form $\phi(\mathcal{L} \cup \mathcal{M})$ where $\mathcal{L}$ and $\mathcal{M}$ are distinct lines containing O. They are shown on the right in Table 1. If $P \neq O$ then $[P]$ is in $\phi(OP \cup \mathcal{M})$ for all lines $\mathcal{M}$ containing O but not P, and so $[P]$ occurs in five such blocks and therefore has overall replication 15, as does $[O]$. Also, the concurrence of $[P]$ and $[O]$ is 5. If Q is a multiple of P other than O and $\pm P$ then $[Q]$ occurs in these five blocks and so the overall concurrence of $[P]$ and $[Q]$ is 5. If P and Q are neither zero nor multiples of each other then the only block of the second type which contains $\{[P], [Q]\}$

is $\phi(OP \cup OQ)$ and so the overall concurrence of $[P]$ and $[Q]$ is 5.

Thus we have a balanced design with number of treatments $v = 13$, replication $r = 15$, number of blocks $b = 39$, block size $k = 5$ and common concurrence $\lambda = 5$ (Table 1).

The subgroup of the automorphism group of the affine plane which fixes the point O is the general linear group $\mathrm{GL}(2,5)$ of all invertible 2×2 matrices over $\mathrm{GF}(5)$. In this group, the elements which fix all the equivalence classes are the identity and scalar multiplication by -1. Therefore the quotient group $\mathrm{GL}(2,5)/\langle -1 \rangle$, which has order 240, is contained in the group of permutations induced on the treatments by the automorphism group of the design.

	$\{O,A,B,C,D\}$
$\{C,E,G,I,K\}$	$\{O,A,B,E,F\}$
$\{D,J,F,L,H\}$	$\{O,A,B,G,H\}$
$\{A,E,J,H,K\}$	$\{O,A,B,I,J\}$
$\{B,G,F,L,I\}$	$\{O,A,B,K,L\}$
$\{A,G,L,J,C\}$	$\{O,C,D,E,F\}$
$\{B,I,H,D,K\}$	$\{O,C,D,G,H\}$
$\{A,I,D,L,E\}$	$\{O,C,D,I,J\}$
$\{B,K,J,F,C\}$	$\{O,C,D,K,L\}$
$\{A,K,F,D,G\}$	$\{O,E,F,G,H\}$
$\{B,C,L,H,E\}$	$\{O,E,F,I,J\}$
$\{A,C,H,F,I\}$	$\{O,E,F,K,L\}$
$\{B,E,D,J,G\}$	$\{O,G,H,I,J\}$
each block above occurs twice	$\{O,G,H,K,L\}$
	$\{O,I,J,K,L\}$

Table 1: Design for 13 treatments

The matrix

$$\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

in $\mathrm{GL}(2,5)$ has order 24 and induces the 12-cycle

$$([E]\ [G]\ [C]\ [J]\ [K]\ [A]\ [F]\ [H]\ [D]\ [I]\ [L]\ [B])$$

on the non-zero equivalence classes. A multiset of orbit representatives under this 12-cycle is

$$\{E, G, C, K, I\} \qquad \text{twice,}$$

$$\{E, G, F, H, O\}, \qquad \{E, K, F, L, O\}, \qquad \{E, J, F, I, O\},$$

and these agree with the initial blocks given in [5].

2.2 Construction from the icosahedron

In the second construction, also given in [1], the treatments are the 12 vertices of the icosahedron and one extra treatment labelled O. For each vertex, there are two blocks which contain the five neighbours of that vertex. Figure 2 shows that these are the 24 blocks on the left of Table 1. For each pair of opposite edges, there is a block containing O and their four vertices: these are the blocks on the right of Table 1.

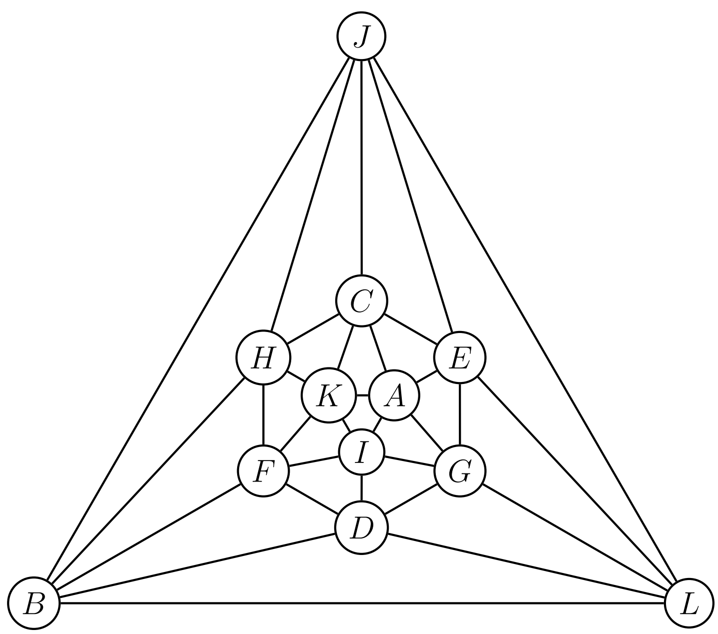

Figure 2: Design for 13 treatments shown on the icosahedron

Clearly the icosahedral group $C_2 \times A_5$ acts on the design. An additional automorphism of order 2 can be derived from the fact that the distance-1 and distance-2 graphs of the icosahedron are isomorphic, and an isomorphism between them preserves the multiset of blocks.

3 A generalisation

We will now give a much more general version of the construction from an affine plane in the preceding section.

The first ingredient in our construction is a balanced incomplete-block design $D_1 = (V_1, B_1)$ with parameters $(v_1, b_1, r_1, k_1, \lambda_1 = 1)$. We will use the terms 'points' and 'lines' for the treatments and blocks of the design D_1. We also require a group Z of automorphisms of D_1, fixing a point O, fixing setwise all the lines containing O, and acting semiregularly on the points different from O (that is, the stabiliser of a point $P \neq O$ is the identity). It follows that the order z of Z is a divisor of $k_1 - 1$. We will assume that $z > 1$; the construction works when $z = 1$ but produces the starting design D_1.

Let $[P]$ denote the Z-orbit of P, and $[\mathcal{L}]$ the Z-orbit of $\mathcal{L}$, where P is any point different from O and $\mathcal{L}$ any line not containing O. The treatments of our new design $D = (V, B)$ are the orbits $[P]$ for $P \neq O$, together with the special point O. The blocks of D not containing O are the orbits $[\mathcal{L}]$ for $O \notin \mathcal{L}$; the block $[\mathcal{L}]$ is incident with the treatment $[P]$ whenever $P \in \mathcal{L}$. We take s copies of each such block, where s is to be determined. Let B' be the multiset of these blocks.

Each line of D_1 containing O is made up of $(k_1 - 1)/z$ orbits $[P]$ together with the point O. To produce a block of k_1 treatments, we select z such lines and take all the corresponding orbits together with O. In order to perform this selection, we need an auxiliary design $D_2 = (V_2, B_2)$ with parameters

$$(v_2 = r_1, b_2, r_2, k_2 = z, \lambda_2).$$

We identify the treatments of D_2 with the lines of D_1 containing O; then, for each block of D_2, we construct as above a block of D containing O. We take t copies of each such block, where t is to be determined. Let B'' be the multiset of such blocks.

We have now constructed our design $D = (V, B)$ with $B = B' \cup B''$, having $v = 1 + (v_1 - 1)/z$ treatments, and having $k = k_1$ treatments in each block. We now examine its properties.

First we consider replication. The treatment O lies in no blocks of B', and in all $|B''| = tb_2$ blocks of B''. For $P \neq O$, the treatment $[P]$ lies in $s(r_1 - 1)$ blocks of B' and in tr_2 blocks of B''. So D is equireplicate if and only if

$$t(b_2 - r_2) = s(r_1 - 1). \tag{1}$$

Next we consider balance. There are three cases to consider. For $P \neq O$, the treatments O and $[P]$ lie in no blocks of B' and in tr_2 blocks of B''. The same is true of the treatments $[P]$ and $[Q]$ if the line PQ contains the point O but P and Q lie in different Z-orbits. Finally, if the line PQ does not contain O, then $[P]$ and $[Q]$ lie in sz blocks of B' and in $t\lambda_2$ blocks of B''. So D is balanced if and only if

$$t(r_2 - \lambda_2) = sz. \tag{2}$$

Now, in the design D_2, we have $zb_2 = r_1 r_2$ and $(z-1)r_2 = \lambda_2(r_1 - 1)$, and so

$$(b_2 - r_2)z = (r_2 - \lambda_2)(r_1 - 1).$$

So equations (1) and (2) are equivalent, and D is balanced if and only if it is equireplicate. Either of the two equations gives the ratio of s to t for which this will be the case.

The design D often inherits automorphisms from D_1 and D_2. Let G be the normaliser of Z in the automorphism group of D_1. Then G fixes O and induces a permutation group $\overline{G}$ on the set of lines of D_1 containing O. Recall that this set is identified with the set of treatments of D_2. Let $\overline{H}$ be the intersection of $\overline{G}$ with the group induced on the treatments by the automorphism group of D_2, and H its inverse image in G/Z. Then H/Z is a group of automorphisms of the design D.

4 Examples

Designs satisfying our requirements for D_1 are not very common. One important class of examples consists of the designs of points and lines in affine spaces $\mathrm{AG}(m,q)$ over finite fields. Such a design has parameters

$$(v_1 = q^m, b_1 = q^{m-1}(q^m - 1)/(q-1), r_1 = (q^m - 1)/(q-1), k_1 = q, \lambda_1 = 1).$$

Taking O to be the origin of the underlying vector space, the group Z_0 of scalar multiplications acts in the prescribed fashion (fixing all lines through O and acting semiregularly on the remaining points). This group is cyclic of order $q - 1 = k_1 - 1$, and has a subgroup Z of each order z which divides $q - 1$, also acting in the required fashion.

The normaliser of any such subgroup Z in the affine group is the group $\Gamma\mathrm{L}(m,q)$ generated by linear transformations and field automorphisms. For simplicity, we consider just the group $\mathrm{GL}(m,q)$ of linear transformations. This acts on the set of lines containing O as the projective general linear group $\mathrm{PGL}(m,q)$.

Rather than simply find any auxiliary design D_2 with the correct block size and take the intersection of its automorphism group with $\mathrm{PGL}(m,q)$, we could choose D_2 to admit the whole of this group, in which case we get

GL$(m,q)/Z$ as a group of automorphisms of D. Note that the designs with arbitrary block size admitting PGL$(2,q)$ have been determined [2, 3].

The group GL(m,q) contains a *Singer cycle* C, a cyclic subgroup permuting the points of AG(m,q) other than O transitively. If we ensure, as above, that GL$(m,q)/Z$ acts on the design D, then since $Z \le C$ we see that C/Z is a cyclic subgroup permuting the treatments of D other than O transitively; that is, D is 1-rotational.

Suppose that q is odd, and take $z = 2$, so that $Z = \{+1, -1\}$. The auxiliary design D_2 should have block size 2; we take the 'pair design' whose blocks are all 2-element subsets of the $(q^m - 1)/(q-1)$ treatments. For balance, we require

$$2(q-1)s = (q^m - 2q + 1)t.$$

So we may take $s = (q^m - 2q + 1)/2(q-1)$ and $t = 1$, to obtain a balanced incomplete-block design with parameters

$$\begin{aligned}(v = (q^m+1)/2, b = (q^m+1)(q^m-1)(q^{m-1}-1)/4(q-1)^2,\\ r = q(q^m-1)(q^{m-1}-1)/2(q-1)^2, k = q, \lambda = q(q^{m-1}-1)/(q-1)).\end{aligned}$$

For $m = 2$ (that is, taking D_1 to be an affine plane), the parameters are

$$((q^2+1)/2, (q^2+1)(q+1)/4, q(q+1)/2, q, q).$$

For $q = 5$, we obtain our motivating example.

In this case, we can take $G = \mathrm{GL}(m,q)$. Since every permutation is an automorphism of the pair design, we have $H = G$, and we see that GL$(m,q)/\{\pm 1\}$ is a group of automorphisms of D.

Here are some further special cases.

1. Consider the case when $m = 2$, $q = 7$ and $z = 3$. Figure 3 shows the affine plane over GF(7) with the points labelled according to their equivalence classes. The auxiliary design is for 8 treatments in blocks of size 3. The smallest possibility is to have all 3-subsets, so $b_2 = 56$ and $r_2 = 21$. Thus we need $35t = 7q$ and so we may take $t = 1$ and $s = 5$. This gives the design in Table 2. This design admits the group GL$(2,7)/Z_3$.

2. When $m = 2$ and $q = 9$ we may take $z = 4$. Now the auxiliary design can be the Steiner system $S(3,4,10)$, which has $b_2 = 30$ and $r_2 = 12$. Hence we can take $t = 1$ and $s = 2$ to obtain a design with parameters

$$(v = 21, b = 70, r = 30, k = 9, \lambda = 12).$$

	0	1	2	3	4	5	6
0	O	A	A	B	A	B	B
1	C	E	G	I	K	M	P
2	C	K	E	M	G	P	I
3	D	N	J	F	Q	L	H
4	C	G	K	P	E	I	M
5	D	J	Q	H	N	F	L
6	D	Q	N	L	J	H	F

Figure 3: The affine plane over GF(7)

$\{C, E, G, I, K, M, P\}$	$\{D, F, H, J, L, N, Q\}$
$\{A, E, G, J, K, N, Q\}$	$\{B, F, H, I, L, M, P\}$
$\{A, D, G, I, L, M, Q\}$	$\{B, C, H, J, K, N, P\}$
$\{A, D, F, I, K, N, P\}$	$\{B, C, E, J, L, M, Q\}$
$\{A, C, F, H, K, M, Q\}$	$\{B, D, E, G, L, N, P\}$
$\{A, D, E, H, J, M, P\}$	$\{B, C, F, G, I, N, Q\}$
$\{A, C, F, G, J, L, P\}$	$\{B, D, E, H, I, K, Q\}$
$\{A, C, E, H, I, L, N\}$	$\{B, D, F, G, J, K, M\}$

each block above occurs five times

the remaining 56 blocks each contain O and three of the following pairs

$\{A, B\}$	$\{C, D\}$
$\{E, F\}$	$\{G, H\}$
$\{I, J\}$	$\{K, L\}$
$\{M, N\}$	$\{P, Q\}$

Table 2: Design for 17 treatments: $q = 7$ and $z = 3$

Since the automorphism group of D_2 is $\mathrm{P\Gamma L}(2,9)$, which is precisely the group induced on the lines through the origin by $\Gamma\mathrm{L}(2,9)$, we can choose the identification between lines through O and treatments of D_2 so that the design D admits $\Gamma\mathrm{L}(2,9)/Z_4$.

3. If $m = 2$ and $z = q-1$, then D_2 has $q+1$ treatments and block size $q-1$, so the blocks are all the $(q-1)$-subsets of V_2. For $z = q-1$ and $m > 2$, this is not the case; the treatments of D_2 are the points of the projective space $\mathrm{PG}(m-1,q)$, and there are various choices for D_2 admitting the group $\mathrm{PGL}(m,q)$. One such choice is to take the blocks to be all $(q-1)$-subsets of lines of $\mathrm{PG}(m-1,q)$. Some examples appear in Table 3 below.

   With this choice, the design has a trivial combinatorial description which works much more generally. The blocks in B' are lines of $\mathrm{PG}(m-1,q)$ with one point removed; the blocks in B'' are lines of $\mathrm{PG}(m-1,q)$ with two points removed and the extra point O adjoined. Adjusting the multiplicities gives a balanced design.

Table 3 lists small designs obtained by applying our construction to an affine geometry of dimension m over $\mathrm{GF}(q)$, taking b_2 to be as small as possible and then s as small as possible for that value of b_2. Here n denotes the size of the group of automorphisms induced by $\mathrm{GL}(m,q)$ guaranteed by the construction. Only one of these designs falls into the catalogue of balanced incomplete-block designs having repeated blocks given by Dobcsányi *et al.* [4]; this is van Lint's design for 13 treatments which was our motivating example (and is discussed in [4, Section 6.2.11]).

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q	m	z	b_2	s	t	v	b	r	k	λ	n
4	3	3	210	9	1	22	1155	210	4	30	60480
4	4	3	3570	41	1	86	76755	3570	4	126	987033600
5	2	2	15	2	1	13	39	15	5	5	240
5	3	2	465	29	2	63	11718	930	5	60	744000
5	3	4	465	27	2	32	5952	930	5	120	372000
7	2	2	28	3	1	25	100	28	7	7	1008
7	2	3	56	5	1	17	136	56	7	21	672
7	3	2	1596	55	2	172	78432	3192	7	112	16892064
7	3	3	532	9	1	115	8740	532	7	28	11261376
9	2	2	45	4	1	41	205	45	9	9	2880
9	2	4	30	2	1	21	70	30	9	12	1440
9	3	2	4095	89	2	365	332150	8190	9	180	169827840
11	2	2	66	5	1	61	366	66	11	11	6600
13	2	2	91	6	1	85	595	91	13	13	13104
16	2	3	680	35	1	86	3655	680	16	120	20400

Table 3: Small designs obtained by applying our construction to affine geometries

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