What is a design? How should we classify them?

R. A. Bailey and Peter J. Cameron
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS, UK

December 18, 2006

If you were to go into a motorcycle-parts department and ask them for a feedback assembly they wouldn't know what the hell you were talking about. They don't split it up that way. No two manufacturers ever split it up quite the same way and every mechanic is familiar with the problem of the part you can't buy because you can't find it because the manufacturer considers it part of something else.

Robert M. Pirsig, Zen and the Art of Motorcycle Maintenance

1 Introduction

In 2001, the United Kingdom Engineering and Physical Sciences Research Council awarded the authors of this paper and Leonard Soicher a grant for "A Web-based resource for design theory". Planning how to put a catalogue of designs on the web forced us to think about the questions which are posed in the title of this paper.

To a mathematician, the definition of a design probably starts like this: "A design consists of a set X of points, and a collection $\mathcal{B}$ of k-element subsets of X called *blocks*, such that ...". (See the book [1], for example.) To a statistician, a design is a rule for allocating treatments to experimental units in, for example, an agricultural field trial or a clinical trial. It is clear that, even if the statistician's definition is turned into mathematics, it looks quite different from the mathematician's definition, and is likely to be much more general.

Further thought shows that an experimental design is in some ways closer to the notion of a *chamber system* [33], a set carrying several partitions related in certain specified ways. The elements are the experimental units; they are partitioned according to the treatments allocated, and also according to "nuisance factors" which the experimenter must allow for, such as fields on different farms, or patients at different hospitals.

Many important chamber systems arise from geometries, so that the chambers are identified with maximal flags in the geometry [8]. In a similar way, a statistician's block design is often the chamber system arising from the mathematician's design (so that the experimental units correspond to incident point-block pairs, and the points to treatments, the blocks to nuisance factors).

There is a further point to note. The mathematician's conditions on a block design are exemplified by the definition of a t-design, in which the dots in the earlier definition say "every set of t points of X is contained in exactly λ blocks". A statistician calls a 2-design a "balanced incomplete-block design", and uses one if it exists, since such designs are best according to various optimality criteria. If, however, the parameters forced on the experimenter by the material available do not correspond to a 2-design, then a weaker condition could be imposed, and there are various choices about how to do this.

The purpose of this paper is to develop a classification scheme which covers as many as possible of the designs used or studied on either side of the divide. We could classify designs according to abstract concepts, according to convenient ways of representing them, or according to what they will be used for. We have tried to adopt a reasonable compromise between these three. Of course, there will have to be many sideways pointers within the classification.

2 Block designs

2.1 What is a block design?

In the Design Theory Resource Server project [12], we adopted the initial compromise of dealing with block designs, while allowing the possibility of including other types later: see [5]. Block designs are the main common ground between mathematical and statistical approaches to designs; and many other types of design, such as Latin squares, can be efficiently represented as block designs.

At its origin in experimental design, a block design consists of a set of plots partitioned into blocks, together with an allocation of treatments to plots (that is, a function from the set of plots to the set of treatments). In other words, it is a set carrying a partition and a function.

The most important thing about the treatment function F is that it gives rise to another partition of the set of plots, two plots α and β being in the same part if $F(\alpha) = F(\beta)$. So the second view of a block design is a set with two partitions. However, there is an asymmetry between the two partitions: the values of F (the names of the parts of the corresponding partition) matter—they are the treatments—while the names of the blocks do not.

A block design can be represented by its *incidence graph*, a bipartite multigraph whose vertices are the parts of the two partitions, where the multiplicity of the edge between the part A of the treatment partition and the part B of the block partition is the number of plots lying in both A and B. (Thus, there is no edge between A and B if no plot in block B gets treatment A.) This construction preserves the symmetry between the two partitions. However, it loses the individual plots, since two plots in the same block receiving the same treatment are now indistinguishable. However, if the design is binary (equivalently, if the multigraph is simple), the plots are just the edges of the graph.

The final representation, as a multiset of multisets, is the one most commonly used by mathematicians (though the multisets are usually sets). It restores the special role of the treatment partition. The 'points' of the design are the treatments (that is, the point set is one part of the bipartition of the graph). Each 'block' is now the multiset of neighbours of a vertex B in the other part of the bipartition; the multiplicity of a point is equal to the multiplicity of the edge joining it to B. Since the same multiset may occur more than once as a block, the blocks form a multiset of multisets in general. If the design is binary, then the blocks form a multiset of sets; and if in addition there are no 'repeated blocks', we have a set of sets.

The original definition of block design is a good starting point for the discussion of randomization and isomorphism. The statistician does not give the treatment function F directly to the experimenter. First (s)he *randomizes* by choosing a random permutation g of the plots which preserves the partition into blocks. The design given to the experimenter is $g \circ F$. From the experimenter's point of view, it is the actual function (also called a plan or layout) $g \circ F$ which is important: it specifies precisely that pig number 42 gets diet C and at some later date pig number 42 can be identified and weighed. But from the statistician's point of view, F and $g \circ F$ are the same design. That is, renaming the blocks, and renaming the plots within blocks, do change the plan but do not change the design.

On the other hand, a design F is *isomorphic* to a design F' if there is a bijection g' from the plots of the former to the plots of the latter which takes one block partition to the other, and a bijection h' from the former treatment set to the latter, such that $F \circ h' = g' \circ F'$. Thus if h is a permutation of the treatments of the first design then $F \circ h$ is isomorphic to F but may not be the same as F.

If we view a block design as a bipartite graph, the asymmetry is now very clear. We are allowed to rename the vertices which represent blocks but we are not allowed to rename the vertices which represent treatments.

The view as a multiset of multisets seems to be the most economical view that captures the idea of sameness while allowing isomorphisms to be represented as bijections between relatively small sets (just the sets of treatments).

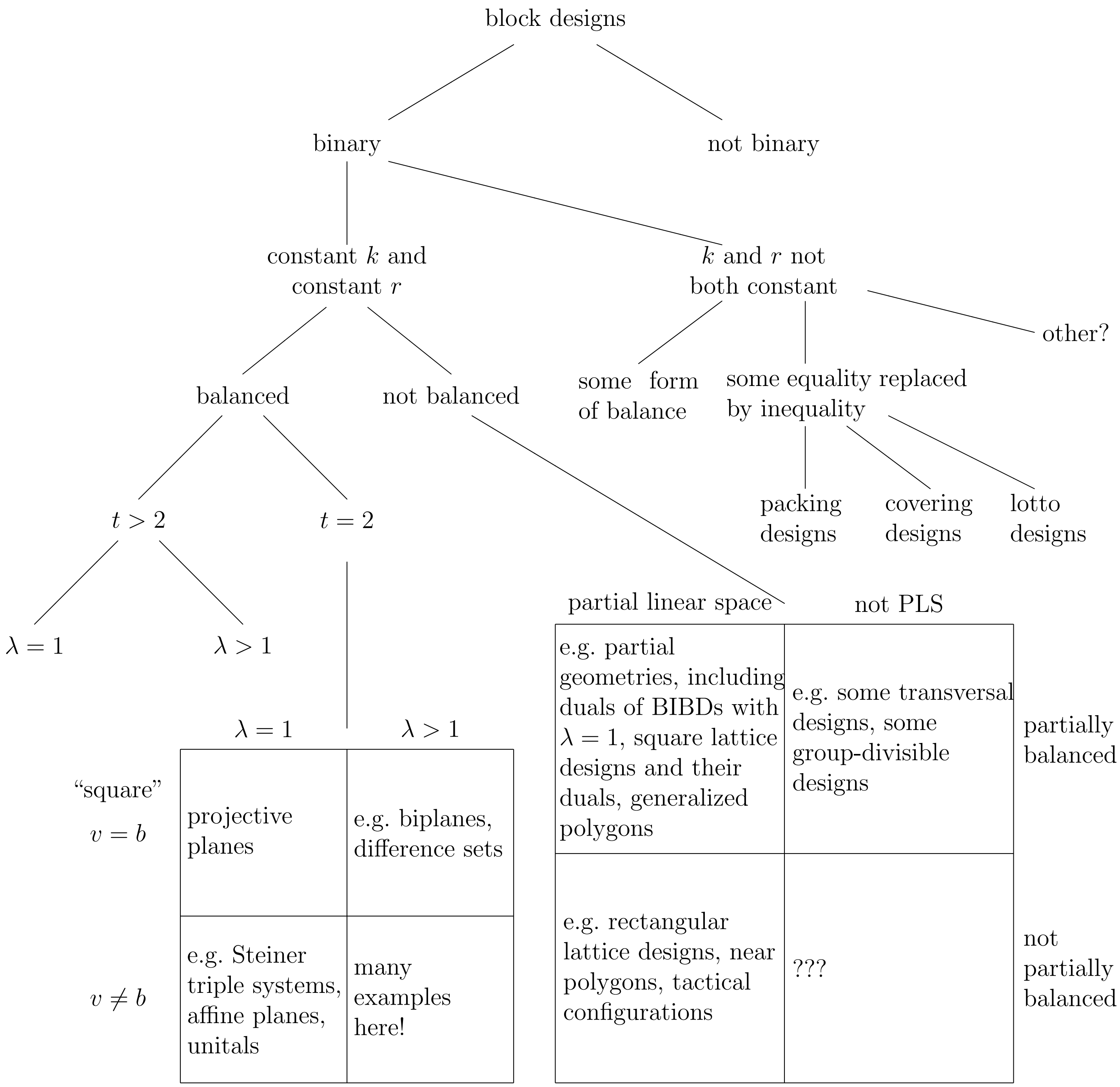
2.2 Classification of block designs

Our classification of block designs is outlined in Diagram B. Here v denotes the number of points (treatments) and b the number of blocks. The first division is into designs that are binary and those that are not. Although we need to be aware of the theory for non-binary block designs, we probably do not need to classify them, because usually they are made in a fairly obvious way from binary designs, for example by adding a complete block to each block, or by doubling up important treatments.

A binary block design is *proper* if each each block contains a fixed number k of points (or treatments), and is *equireplicate* if each point is contained in a fixed number r of blocks. We have split binary designs according to whether or not r and k are both constant: in the former case, combinatorial and statistical properties tend to agree, while in the latter case they do not. We need to be careful and generous with terminology here: what the statistician calls an equireplicate proper design a hypergraph theorist calls a regular uniform hypergraph and a combinatorial design theorist calls a 1-design.

We have further split 1-designs into those that are balanced and those that are not. Here *balanced* means a t-design for some t with $t \geq 2$. Recall that, in a t-(v, k, λ) design, there are v points and b blocks of size k, and any set of t points is contained in exactly λ blocks. The widely-used name 't-design' was introduced in print by Hughes [17], although he claims that it was his colleague D. G. Higman who suggested '2-design'.

Within balanced designs, the distinctions between $\lambda = 1$ (Steiner systems) and $\lambda > 1$ is important. For $t = 2$, so is the distinction between square and non-square designs. We follow Hughes and Piper [18] in calling a block design



block designs
binary
not binary
constant k and constant r
k and r not both constant
balanced
not balanced
some form of balance
some equality replaced by inequality
other?
packing designs
covering designs
lotto designs
$t > 2$
$t = 2$
$\lambda = 1$
$\lambda > 1$
partial linear space
not PLS
$\lambda = 1$
$\lambda > 1$
"square" $v = b$
$v \neq b$
projective planes
e.g. biplanes, difference sets
e.g. Steiner triple systems, affine planes, unitals
many examples here!
e.g. partial geometries, including duals of BIBDs with $\lambda = 1$, square lattice designs and their duals, generalized polygons
e.g. some transversal designs, some group-divisible designs
partially balanced
e.g. rectangular lattice designs, near polygons, tactical configurations
???
not partially balanced

Diagram B

square if it has the same number of points as blocks. Square designs are also referred to as "symmetric", though no symmetry between points and blocks is implied. Note that only trivial t-designs with $t > 2$ can be square. A square 2-design with $\lambda = 1$ is a projective plane.

For the non-balanced case, the range of designs which have been studied is much larger. Among these which have been studied are:

- *Partially balanced designs*, those for which there is an association scheme [4] on the set of points such that the *concurrence* of a pair of points (the number of blocks containing the two points) depends only on the associate class containing them. This class includes many which are important in finite geometry, such as generalized polygons, partial geometries, and partial geometric designs. See [11, Chapters IV.21, IV.34], [7], and many other references.

- *Partial linear spaces*, those for which any two points are incident with at most one block. This class overlaps with the first in that it includes generalized polygons and partial geometries; it also contains tactical configurations [11, Chapter IV.6] and SOMAs (simple orthogonal multi-arrays) [32], though the latter have more structure and are better classified elsewhere. Also, partial linear spaces are more general objects; there is no reason to assume that r and k are constant for such designs.

We have cross-classified the designs according to whether or not they have one or other of these two features.

Finally, we come to the situation where r and k are not assumed constant. Here we can do little more than point to some interesting classes. Among these are:

- Designs with some form of balance, such as pairwise balance [1, Section 1.4], variance balance [19, Chapter 2], or efficiency balance [19, Chapter 2], or designs which satisfy some variant of group-divisibility: see [4, Section 1.1] and [1, Section 4.7].

- Designs in which some equality holding in t-designs is replaced by an inequality, especially those which are extremal. Among these are *packing and covering designs*, those in which any t points are contained in at most (resp. at least) λ blocks, which could be regarded as "approximations" to t-designs: see [11, Chapters IV.8, IV.33] and [21]. More generally, *lotto designs* [22] fit here.

In general, these structures are more often regarded as hypergraphs or set systems than as designs.

The various 'balance' attributes which interest the pure mathematician contrast with those of efficiency and optimality which interest a statistician. For a binary, proper design with block size k, the *information matrix* is equal to $R-k^{-1}\Lambda$, where Λ is the matrix of concurrences and R the diagonal matrix of replications. Most optimality criteria are convex symmetric functions of the eigenvalues of this matrix, excluding the zero eigenvalue on the all-1 vector. See [31] for details.

3 Towards a classification

3.1 More general designs

Our general classification starts in Diagram A. The first dichotomy has been introduced in Section 2.1: between two or more sets with incidence relations between them, and a single set with partitions or functions defined on it.

The distinction between partitions and functions is that the *values* of the function matter. For example, a design for five varieties of sunflower is probably equally good no matter what the varieties are, but for a design to fit a polynomial model for response to quantity x of chemical then you need to choose which quantities x to use; this choice is part of the design.

The 'functions' leaf of this tree leads to designs for continuous variables. Although there is a considerable body of research on such designs (see [2, 30]), we do not consider them any further in this paper.

The 'partitions' branch is split according to the number of partitions on the set. A set with two partitions is essentially a block design, as discussed in Section 2.1. Three or more partitions give many more possibilities, some of which are touched on in Section 4 and Diagram C.

The 'incidence' branch in split into families of sets and multipartite graphs. Each of these has one branch leading to block designs, as discussed in Section 2.1. The other three are described below.

3.2 A note on diagrams

Diagrams are used to describe chamber systems and geometries [8], and in this paper we will use diagrams to represent designs in both the incidence and partition settings. However, we must make clear that the diagrams mean quite different things.

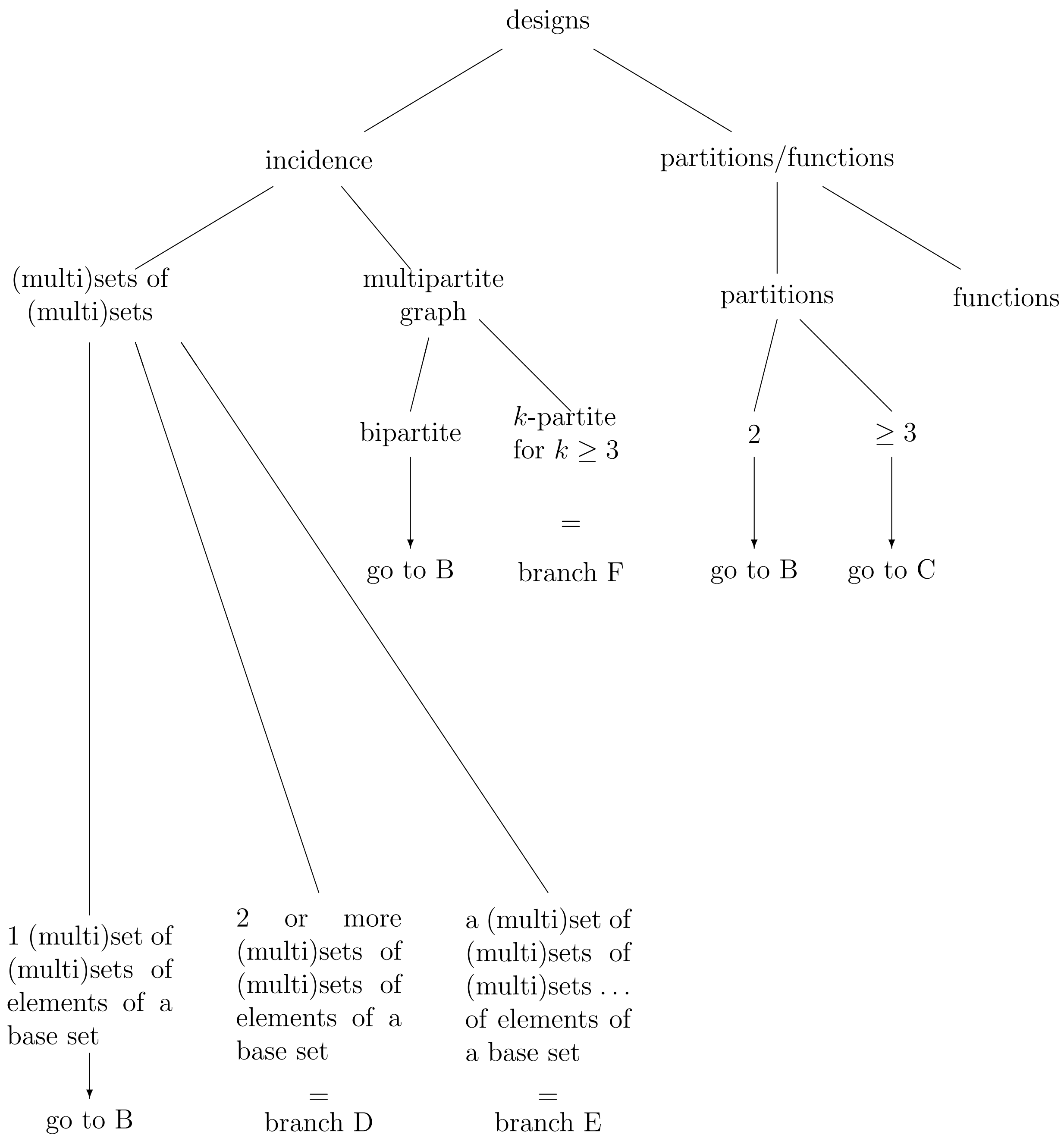

designs
incidence
partitions/functions
(multi)sets of (multi)sets
multipartite graph
partitions
functions
bipartite
k-partite for $k \geq 3$
2
≥ 3
go to B
=
branch F
go to B
go to C
1 (multi)set of (multi)sets of elements of a base set
2 or more (multi)sets of (multi)sets of elements of a base set
a (multi)set of (multi)sets of (multi)sets . . . of elements of a base set
go to B
=
branch D
=
branch E

Diagram A

When we have a family of sets with an incidence relation between each pair of sets, we represent the design by a diagram in which the nodes represent the sets in the family, and the label on the edge joining two nodes represents the type of design formed by the incidence structure made up of these two sets only. Similarly, if a design is represented by a family of partitions of a set, we represent it by a diagram in which the vertices correspond to partitions, and the label on the edge joining two vertices represents the type of design formed by just these two partitions. We call these diagrams *global diagrams*.

The diagrams representing a Buekenhout geometry or a chamber system will be called *local diagrams*. Their definition is more subtle, and will be described briefly when we describe these structures.

We will attempt to distinguish the two types of diagram by using solid circles for nodes of global diagrams, and open circles for nodes of local diagrams.

3.3 Designs with several kinds of block

Branch D is like block designs except that there are two or more types of block but all are sub(multi)sets of the same base set. Examples include affine and projective geometry (blocks are lines, planes etc), rectangular lattice designs for $n(n-1)$ treatments (the subsets are spokes and fans, see [6]), and the sort of algebraic geometry where lines and conics (or more general algebraic varieties) count as blocks. Group-divisible designs can be included here: the subsets are blocks and groups. One *could* consider resolved block designs in this branch, with one type of block for each replicate, but they probably are more conveniently handled in the next branch.

Graph decompositions [11, Chapter IV.22] fit into this branch. A decomposition of a complete graph into copies of a fixed graph G can be regarded as a set (the set of edges of the complete graph) with two families of subsets: the stars (these define the complete graph structure), and the copies of G.

3.4 Iterated block designs

Branch E is really iterated block designs. It includes (i) resolved block designs, which are replicates consisting of blocks consisting of treatments; (ii) nested block designs, which are large blocks consisting of small blocks consisting of treatments (for example, see [3, 24]); (iii) whist tournaments, which are rounds consisting of tables consisting of pairs consisting of players [1, Chapter 11]. It also includes so-called 'large sets' of block designs or of Steiner triple systems [20].

3.5 Geometries

Branch F includes Buekenhout geometries. A *geometry*, or *Buekenhout geometry*, consists of a set V of objects called *varieties*, a symmetric and reflexive *incidence relation* $*$ on V, and a surjective *type function* from V to a finite set Δ of *types*, satisfying several axioms introduced below. A *flag* is a set of varieties, any two of which are incident; it is called a *transversal flag* if it contains exactly one variety of each type. We require:

(G1) Any maximal flag is transversal.

(G2) Any non-maximal flag is contained in at least two maximaml flags.

Axiom (G1) implies that two vertices of the same type which are incident must be equal, and that the restriction of the type function to a flag is injective. We can think of a geometry as a multipartite graph (with a loop at each vertex), where the parts of the multipartition are labelled by types; a maximal clique contains one vertex of each type.

Let F be a flag. Its *type* is its image under the type function, and its *cotype* is the complement of the type (in Δ). The *residue* of F consists of all varieties not in F but incident with every member of F; it is a geometry (satisfying (G1) and (G2)) over the cotype of F. We call the cardinalities of the type and cotype the *rank* and *corank* of F respectively.

A geometry is *connected* if the corresponding multipartite graph is connected. A geometry is *residually connected* if the residue of any flag of corank at least 2 is connected. It is usual to require this:

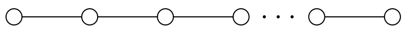
(G3) The geometry is residually connected.

Now, for all distinct types i and j, let $\mathcal{G}_{ij}$ be a class of rank 2 geometries (whose elements are called *points* and *blocks*), with the property that $\mathcal{G}_{ji}$ consists of the duals of the geometries in $\mathcal{G}_{ij}$. A *diagram* D consists of the type set Δ with an assignment of classes of rank 2 geometries to each pair of vertices subject to the above condition. Now a geometry is said to *belong to the diagram* D if the residue of any flag of cotype $\{i, j\}$ belongs to $\mathcal{G}_{ij}$, when varieties of types i and j are identified with points and blocks respectively.

In this way, a diagram specifies local information about a geometry (the structure of rank 2 residues). These diagrams are "local" in the sense outlined earlier.

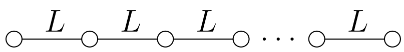
A diagram is a very concise way to exhibit axioms for a class of geometries. We illustrate with just two examples. We represent the class of geometries in which any point and block are incident by the absence of an edge between corresponding types; a single unadorned edge denotes a projective plane; and an edge with label L denotes a linear space (with points represented by the left-hand vertex).

1. A geometry with diagram



   is a projective space whose dimension is equal to the number of nodes.

2. A geometry with diagram



   is the geometry of flats (closed subspaces) of a matroid with rank one greater than the number of nodes. In particular, perfect matroid designs [13] fit here.

We have seen that an incidence structure with two types of varieties can often be represented efficiently by taking blocks to be subsets of the set of points. This can sometimes be extended to geometries of higher rank. We select one type, and call the varieties of this type "points". The *shadow* of a variety is now the set of points incident with it. In some cases (including those in the above results, with points chosen to correspond to the left-hand end of the diagram), two varieties are incident if and only if the shadow of one is contained in the shadow of the other; so the geometry can be conveniently represented as several families of subsets of the point set.

Among the important classes of Buekenhout geometries are projective, affine and polar spaces [9].

Graph decompositions can also be treated here. A decomposition of a complete graph into copies of the graph G is a geometry with three types of varieties (vertices, edges and copies of G) with the obvious incidence relations between them.

4 Designs with several factors

In Branch C each object consists of a set with three or more partitions on it. These are known as 'multi-factor designs' because statisticians use the word 'factor' for both partitions and functions. Hence it is possible for some factors to be equal as partitions even though their names are different.

4.1 The partition lattice

A partition is called *uniform* if all its parts have the same size. Partition P *nests* partition Q if every part of Q is contained in a part of P. Given partitions P and Q, their *supremum* is the finest partition that nests both of them, and their *infimum* is the coarsest partition that is nested in both of them. There are two trivial partitions: the universal partition U, with a single part, and the *equality* partition E, whose parts are singletons.

For the purpose of this paper, we temporarily define a collection of partitions on a single set to be *supreme* if the supremum of any two is either the universal partition or included in the set, to be *flat* if the supremum of any two inequivalent ones is the universal partition, and to be *nested* if it is not flat but the supremum of any two partitions is the universal partition unless one is nested in the other.

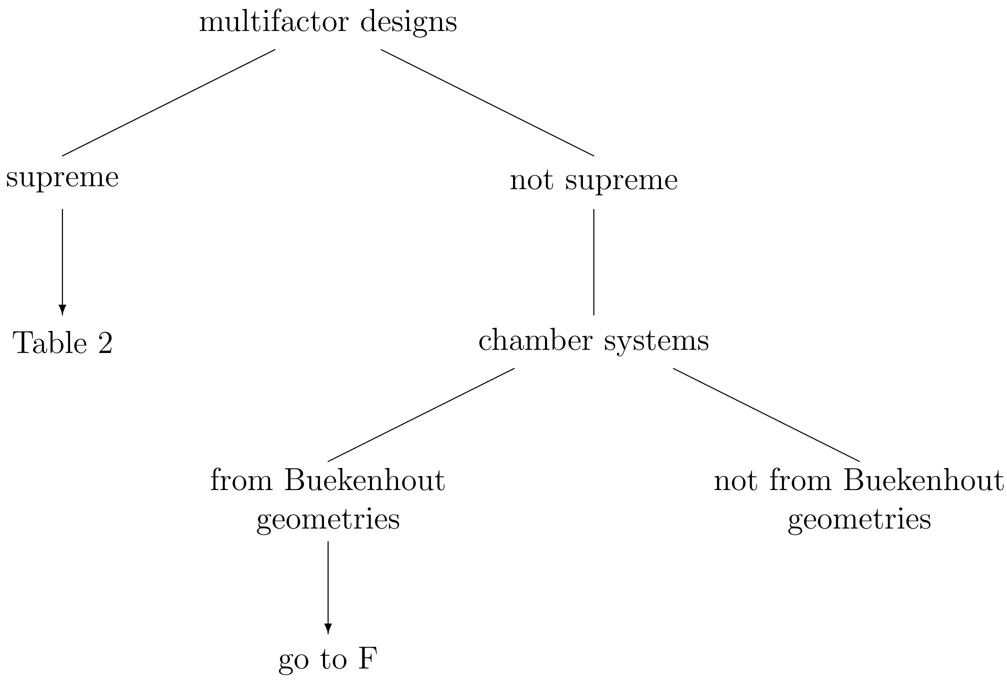
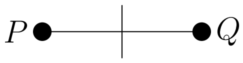
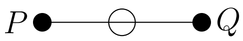
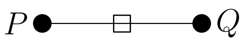
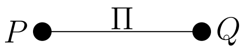
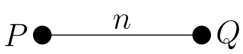


Diagram C

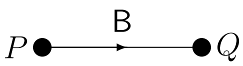
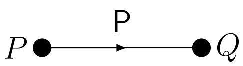
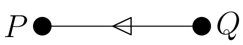
Diagram C splits multifactor designs according to whether they are supreme or not. The two types are discussed further in Sections 4.2 and 4.4.

Many designs are defined by pairwise relations on the partitions. Some of these are shown diagrammatically in Table 1. Partition P is *orthogonal* to partition Q if their averaging matrices commute with each other; *strictly orthogonal* if they are orthogonal and their supremum is U. (Some authors call these ‘geometrically orthogonal’ and ‘orthogonal’ respectively.) Thus an orthogonal array of strength two is simply a collection of uniform partitions which are pairwise strictly orthogonal.

Other relations can be specified by properties of the block design defined by P and Q as in Section 2.1. Thus symmetric relations include orthogonality; strict orthogonality; having the incidence of a partial linear space (what a statistician calls a $\{0,1\}$-design); having the incidence of a square balanced block design; and having the incidence of a generalized polygon. Non-symmetric relations include nesting (which is a special case of orthogonality) and non-square balanced block designs.

	P is orthogonal to Q
	P is strictly orthogonal to Q
	P is a square balanced block design with respect to Q
	P is a partial linear space with respect to Q
	P is a generalized n-gon with respect to Q

(a) symmetric relations

	P is balanced with respect to Q but Q is not balanced with respect to P
	P is partially balanced with respect to Q and we do not care about Q with respect to P
	P is nested in Q and not equivalent to Q

(b) non-symmetric relations

Table 1: Some relations between pairs of partitions

4.2 Chamber systems

In his ground-breaking work on geometry, Tits originally took a geometric approach like that later developed by Buekenhout, but switched later to an approach based on partitions, which we now describe.

A *chamber system* consists of a set Ω of objects called *chambers*, and a finite collection $\{\pi_i : i \in \Delta\}$ of non-trivial partitions of Ω, having the properties

(C1) the infimum of any two of the partitions π_i is the partition E into singletons;

(C2) the supremum of all the partitions is the partition U with a single part.

The second axiom is called *chamber-connectedness.*

The rank of the chamber system is $|\Delta|$.

Any geometry (in the sense of Buekenhout) gives rise to a chamber system as follows:

- the chambers are the maximal (transversal) flags;
- two maximal flags F and F' are *i-adjacent* (that is, lie in the same part of π_i) if they contain the same varieties of each type $j \neq i$.

Chamber-connectedness of the chamber system implies connectedness of the geometry, and is implied by residual connectedness; neither of these implications reverses.

Not every chamber system comes from a geometry in this way. In a chamber system which does come from a (residually connected) geometry, two maximal flags share a variety of type i if and only if they are connected by a path using only adjacencies of types different from i. So we can identify varieties with connected components of this graph. Two varieties of different types are incident if and only if the corresponding connected components have non-empty intersection.

Now let L be a Latin square, Ω the set of its cells, and let π_1, π_2 and π_3 be the partitions of Ω defined by 'same row', 'same column', and 'same symbol' respectively. Then $(\Omega, \pi_1, \pi_2, \pi_3)$ is a chamber system. The graph formed by any two of these partitions is connected (it is a square grid), so it is not possible to construct varieties as connected components.

A chamber system of rank 2, however, does arise from a geometry. So we can say that a chamber system *belongs to the diagram D* if, for any two types i and j, every connected component of the graph formed by i- and j-adjacency belongs to the class $\mathcal{G}_{ij}$ of rank 2 geometries (or chamber systems).

Thus, a Latin square (as chamber system) belongs to the diagram consisting of three pairwise non-adjacent vertices.

We refrain from developing further the fascinating theory of geometries and chamber systems; see, for example, [26].

When a chamber system comes from a geometry, it is probably more efficient to consider it as such. (See the comments on Buekenhout geometries in the above discussion of Branch F.) For example, consider the 3-dimensional projective geometry of order 2: as a set of sets it has 15 points; as a geometry it has 65 varieties; and as a chamber system it has 315 chambers.

4.3 Some common statistical designs

In a factorial design, the set of treatments is cross-classified by two or more factors, whose values specify the treatments uniquely. Classically, the treatments are identified with the elements of an Abelian group in such a way that the factors are the characters which canonically generate the dual group. Then a *confounded* factorial block design with replication 1 is specified by identifying the blocks with the cosets of a chosen subgroup: see [15].

In a fractional factorial design, not all of the potential treatments are applied, so the design is a subset of the treatments. Partitions which are equal on the chosen subset are said to be *aliased*. So-called *regular* fractions are those in which partitions of interest are either aliased or orthogonal to each other. An extreme case are the *Plackett–Burman* designs, in which the partitions given by the original treatment factors are uniform and pairwise strictly orthogonal, while any infima are ignored. Of course, these are the same as orthogonal arrays of strength two.

A resolved block design with r replicates can be considered as a family of r partitions of the set of treatments. When $r = 2$, this identifies the design with a smaller square design, as was fruitfully exploited in [34]. When the partitions are pairwise strictly orthogonal, the block design is *affine resolved*.

There are many experiments where the plots form a rectangle, either in space, as in a field experiment, or abstractly, as in a clinical trial, where the rows and columns are time-periods and patients respectively. Designs for such experiments are generally called row-column designs. Typically, the rows and columns are strictly orthogonal to each other, and treatments are allocated to plots to satisfy some conditions. The classic example is a Latin square, which is a special case of an orthogonal array of strength two. Some others are given in Section 4.4. More complex still are experiments in blocks where each block is a rectangle: these are called nested row-column designs.

4.4 Supreme designs

Supreme multi-factor designs are also local in the sense that pairwise relations between the partitions are required to hold only within each part of their supremum. However, because we require suprema to be in the set of listed partitions, we usually impose some conditions on these coarser partitions as well as on the finer ones.

	Flat	Nested	Other
pairwise relations suffice	orthogonal arrays of strength 2; semi-Latin squares; Plackett-Burman designs (possibly in blocks); Youden squares; Youdenized square non-balanced IBDs; double Youden rectangles; other designs in 2 rectangular replicates	SOMAs; Howell designs; Room squares; balanced tournament designs	orthogonal arrays of strength ≥ 3; split-plot designs; confounded factorial designs; regular fractional factorial designs;
pairwise relations do not suffice	Preece triples; Freeman-Youden rectangles; triple arrays; other generally balanced row-column designs; designs in r rectangular replicates for $r \geq 3$		nested block designs: go to E; other nested row-column designs; other fractional factorial designs

Table 2: Cross-classification of supreme multi-factor designs with uniform factors, showing some examples

Supreme multi-factor designs in which all partitions are uniform are cross-classified in Table 2. One classification is into flat, nested and other. The second classification is by whether or not the structure can be defined by pairwise relations between the partitions.

So long as each partition is orthogonal to all the others, or to all but one of the others, the pairwise relations seem to be enough to define the design. Table 3 gives some examples.

An $(n \times n)/k$ semi-Latin square [35] is an orthogonal array of strength two with three partitions: one has n rows of size nk, one has n columns of size nk, the third has nk letters of size n. A simple orthogonal multi-array (SOMA) is a semi-Latin square with an extra condition. Now we need to consider the infimum of the row partition and the column partition: call its parts *cells*. For a SOMA, the incidence of letters in cells is that of a partial linear space.

By Hall's Marriage Theorem, the plots of any square block design for b treatments replicated k times can be arranged as a $b \times k$ rectangle: the rows are the blocks, and each treatment occurs once in each column. If the block design is balanced, this row-column design is called a *Youden square* [28], but any square block design can be 'Youdenized' in this way.

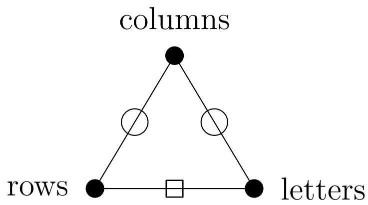
If the b treatments in a Youden square are Latin letters, it may be possible to superimpose k Greek letters so that they are strictly orthogonal to rows and to Latin letters and form a balanced non-binary block design with respect to columns. This is a *double Youden rectangle* [28].

Just as a resolved block design can be considered as a set of r partitions of the treatment set, a resolved design in two rectangular $n \times m$ replicates can be considered as four partitions of the set of treatments: two with n parts of size m, two with m parts of size n. If each is strictly orthogonal to the two of the other kind, we have a useful generalization of a double Youden rectangle.

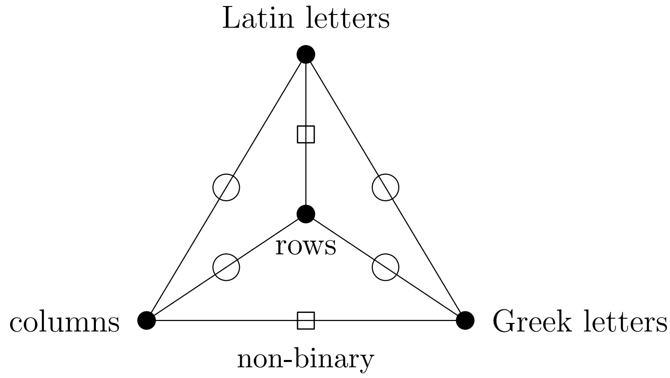
Room squares and Howell designs are like SOMAs except that some of the cells may be empty. In a Room square [1, Chapter 10], the incidence of letters in cells is that of a balanced block design with block size two. In a Howell design, it is that of a partial linear space with block size two. A balanced tournament design [1, Chapter 10] has n rows, $2n-1$ columns and all cells of size two. There are $2n$ letters. Letters are orthogonal to columns, and form a balanced incomplete-block design with cells. The incidence of letters in rows is a non-binary block design in which each row contains two letters once, the others twice.

Multi-factor designs which are neither supreme nor nested but in which every pair of partitions is orthogonal include split-plot designs, counfounded factorial designs and regular fractional factorial designs.

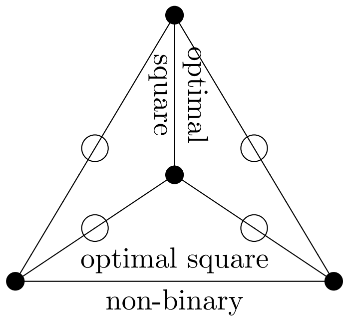
Sometimes the pairwise relations are enough to define the design even



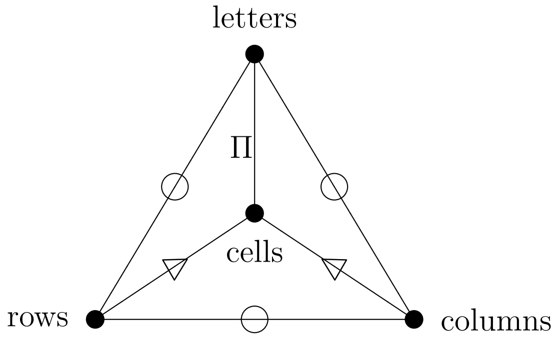
Youden square



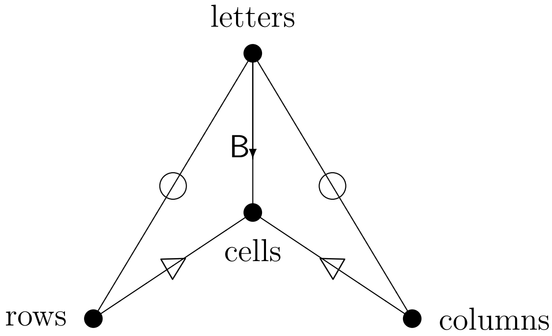
double Youden rectangle



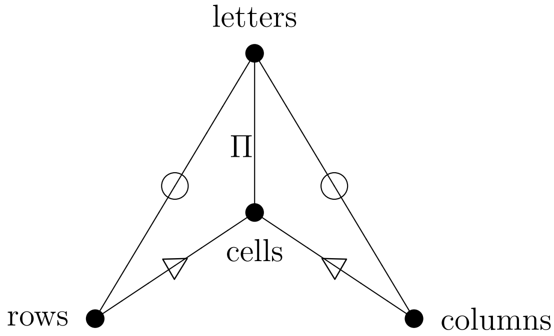
generalization of double Youden rectangle



SOMA: simple orthogonal multi-array



Room square

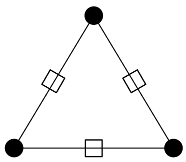


Howell design

Table 3: Some designs defined by pairwise relations on uniform partitions

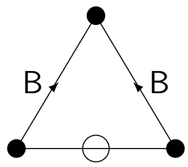
without orthogonality. For example, partitions into n parts of size 2 are just 1-factors of the complete graph K_{2n}. If we insist that each pair of these have the incidence of a generalized n-gon, then any collection of $2n-1$ such partitions forms a perfect 1-factorizion of K_{2n}.

Otherwise,the pairwise conditions do not suffice. Statisticians need extra conditions such as ‘general balance’ [25] or ‘adjusted orthogonality’ [14] or ‘overall balance’ [27]. These are too technical to define here, but can all be expressed both in terms of incidence matrices and in terms of angles between subspaces. Table 4 gives some examples. If all but one of the partitions form a nested chain (as in nested block designs), it is probably more efficient to use the representation from Branch E.



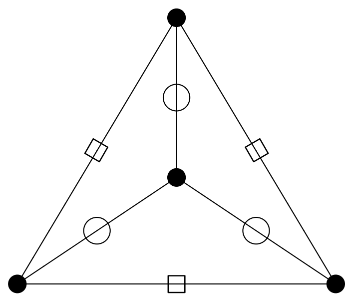
with the extra condition
of overall balance

Preece triple



with the extra condition
of adjusted orthogonality

triple array



with the extra condition
of overall balance

Freeman-Youden rectangle

Table 4: Some designs defined by uniform partitions where the pairwise relations do not suffice

A Preece triple consists of three partitions with b parts of size k, each pair of which have the incidence of a square balanced incomplete-block design. The extra condition needed is ‘overall balance’. If a partitition with k parts of size b can be superimposed on a Preece triple, orthogonal to the three original partitions, we obtain a Freeman-Youden rectangle [29]. A *triple array* [23] is a row-column design in which both rows and columns form a balanced incomplete-block design with respect to letters. The extra condition needed here is 'adjusted orthogonality'.

5 Some further considerations

There is some extra structure that seems to be possible throughout, so we have not incorporated it in any one place. This might be structure within each part of a partition: for example, a directed or undirected circuit in each block (for the Oberwolfach problem or for one version of whist tournaments). It might be extra structure on the set of parts of a partition: for example, an association scheme on the treatments (for partially balanced designs in general). It might be extra structure that can only conveniently be described on the underlying set: for example, row and column neighbours in a square, or an association scheme on the set of plots.

A classification of designs needs to concern itself with structures that are not themselves designs. Most important, perhaps, are permutation groups, which arise as subgroups of the automorphism groups of designs. There has been considerable interest in, for example, block designs which are *cyclic* (admitting a transitive cyclic group of automorphisms) or 1-*rotational* (admitting a cyclic group fixing a point and transitive on the remainder). A large automorphism group enables the design to be presented more concisely: we can give generators for the group and a set of base blocks (orbit representatives for the blocks).

Partially balanced designs involve association schemes in a similar way. There is no concise description of an association scheme (again, unless it has a large automorphism group). The catalogue by Hanaki and Miyamoto [16] describes an association scheme by a matrix whose (i, j) entry is k if the pair $\{i, j\}$ lies in the kth associate class.

Each type of design may be used in one or more ways. The appropriate concepts of 'same', of automorphism and of randomization will depend on the use. (For example, think of the five uses of a Latin square given in the *Encyclopaedia of Design Theory* [10].) Some authors regard different uses of a Latin square as being different designs.

References

[1] I. Anderson, *Combinatorial Designs and Tournaments*, Oxford University Press, Oxford, 1997.

[2] A. C. Atkinson and A. N. Donev, *Optimum Experimental Designs*, Oxford University Press, Oxford, 1992.

[3] R. A. Bailey, Choosing designs for nested blocks, *Listy Biometryczne 36* (1999), 85–126.

[4] R. A. Bailey, *Association Schemes: Designed Experiments, Algebra and Combinatorics*, Cambridge University Press, Cambridge, 2004.

[5] R. A. Bailey, P. J. Cameron, P. Dobcsányi, J. P. Morgan and L. H. Soicher, Designs on the Web, *Discrete Math.* **306** (2006), 3014–3027.

[6] R. A. Bailey and T. P. Speed, Rectangular lattice designs: efficiency factors and analysis, *Annals of Statistics* **14** (1986), 874–895.

[7] A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, in: *Enumeration and Design*, Academic Press, Toronto, 1984, pp. 85–122.

[8] F. Buekenhout, Diagrams for geometries and groups, *J. Combinatorial Theory* (A) **27** (1979), 121–151.

[9] P. J. Cameron, *Projective and Polar Spaces*, QMW Maths Notes **13**, Queen Mary and Westfield College, London, 1991; second edition available at
`http://www.maths.qmul.ac.uk/~pjc/pps/`

[10] P. J. Cameron (editor), *Encyclopaedia of Design Theory*,
`http://designtheory.org/library/encyc/`

[11] C. J. Colbourn and J. H. Dinitz (editors), *Handbook of Combinatorial Designs* (1st edition), Chapman & Hall/CRC, 1996.

[12] Design Theory Resource Server, `http://designtheory.org/`

[13] M. Deza, Perfect matroid designs, in *Matroid Applications* (ed. N. White), pp. 54–72, Encyc. Math. Appl. **40**, Cambridge University Press, 1992.

[14] J. A. Eccleston and K. G. Russell, Connectedness and orthogonality in multi-factor designs, *Biometrika* **62** (1975) 341–345.

[15] R. A. Fisher, The theory of confounding in factorial experiments in relation to the theory of groups, *Annals of Eugenenics* **11** (1942), 341–353.

[16] A. Hanaki and I. Miyamoto, Classification of association schemes with small vertices, `http://kissme.shinshu-u.ac.jp/as/`

[17] D. R. Hughes, Combinatorial analysis: t-designs and permutation groups, in *Proceedings of the Symposium in Pure Mathematics* 6, American Mathematical Society, Providence, Rhode Island, 1962, pp. 39–41.

[18] D. R. Hughes and F. C. Piper, *Design Theory*, Cambridge University Press, Cambridge, 1985.

[19] J. A. John and E. R. Williams, *Cyclic and Computer-Generated Designs*, Chapman and Hall, London, 1995.

[20] G. B. Khosrovshahi and B. Tayfeh-Rezaie, Large sets of t-designs through partitionable sets: a survey, *Discrete Math.* **306** (2006), 2993–3004.

[21] La Jolla Covering Repository, `http://www.ccrwest.org/cover.html`

[22] P. C. Li and G. H. J. van Rees, Lotto design tables, *J. Combinatorial Designs* **10** (2002), 335–359.

[23] J. P. McSorley, N. C. K. Phillips, W. D. Wallis and J. L. Yucas, Double arrays, triple arrays and balanced grids, *Designs, Codes and Cryptography* **35** (2005), 21–45.

[24] J. P. Morgan, D. A. Preece and D. H. Rees, Nested balanced incomplete block designs, *Discrete Mathematics* **231** (2001), 351–389.

[25] J. A. Nelder, The combination of information in generally balanced designs, *Journal of the Royal Statistical Society, Series B* **30** (1968), 303–311.

[26] A. Pasini, *Diagram Geometries*, Oxford University Press, Oxford, 1994.

[27] D. A. Preece, Non-orthogonal Graeco-Latin designs, in *Combinatorial Mathematics IV* (eds. L. R. A. Casse and W. D. Wallis), Lecture Notes in Mathematics 560, Springer, Berlin, 1976, pp. 7–26.

[28] D. A. Preece, Fifty years of Youden squares: a review, *Bulletin of the Institute of Mathematics and its Applications* **26** (1990), 65–75.

[29] D. A. Preece and N. C. K. Phillips, A new type of Freeman-Youden rectangle, *Journal of Combinatorial Mathematics and Combinatorial Computing* **25** (1997), 65–78.

[30] F. Pukelsheim, *Optimal Design of Experiments* Wiley, New York, 1993.

[31] K. R. Shah and B. K. Sinha, *Theory of Optimal Designs*, Springer, New York, 1989.

[32] L. H. Soicher, SOMA Update `http://www.maths.qmul.ac.uk/~leonard/soma/`

[33] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Math. **382**, Springer-Verlag, Berlin, 1974.

[34] E. R. Williams, H. D. Patterson and J. A. John, Resolvable designs with two replications, *Journal of the Royal Statistical Society, Series B* **38** (1976), 296–301.

[35] F. Yates, Complex experiments, *Journal of the Royal Statistical Society, Supplement* **2** (1935), 181–247.