Automorphism groups of homogeneous structures

Peter J. Cameron

Queen Mary University of London

p.j.cameron@qmul.ac.uk

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Summary

This will be a quick tour round a class of interesting groups which arise as the automorphism groups of countable homogeneous subgroups.

The most famous of these are the automorphism group of the countable "random graph", and the isometry group of the Urysohn metric space.

There is a lot that we don't know about these groups!

B-groups

A group X is a *B-group* if every primitive permutation group G containing X as a regular subgroup is doubly transitive.

(Here G is *primitive* if there is no non-trivial G-invariant equivalence relation, and *doubly transitive* if there are no non-trivial G-invariant binary relations at all. So X is a B-group if, whenever we add permutations destroying all the X-invariant equivalence relations, the resulting group is doubly transitive.)

The name (due to Wielandt) comes from

Theorem: A finite cyclic group of composite order is a B-group. (Burnside)

Finite B-groups

The study of B-groups by Schur and others led to the development of the theory of Schur rings, and connections between permutation groups and representation theory. This is outlined in Wielandt's book.

Using the Classification of Finite Simple Groups, it can be shown that, for almost all n (all except a set of zero density), every group of order n is a B-group.

This is not the same as saying that almost all finite groups are B-groups. It seems that most finite groups have prime power order; and there are many non-B-groups of prime power order, for example elementary abelian groups of order p^n where p is odd or $p^n - 1$ is composite.

Problem: Are there any infinite B-groups?

Infinite B-groups

G. Higman: Let X be a countable group in which every non-identity element has only finitely many square roots. Then X is not a B-group.

Cameron and Johnson: There is a primitive, not doubly transitive group G of countable degree such that every group satisfying Higman's condition (or indeed, a more general condition, given on the next slide) is a regular subgroup of G.

The group G is the automorphism group of the countable *random graph*, see later.

The condition

A *square-root set* in the group X is a set of the form

$$\sqrt{a} = \{x \in X : x^2 = a\}.$$

It is *non-principal* if $a \neq 1$.

The sufficient condition for X to be a regular subgroup of the group G of the last slide is as follows:

X cannot be written as the union of a finite number of translates of non-principal square-root sets together with a finite set.

The proof shows that, if X satisfies this condition, then a random Cayley graph for X is almost surely isomorphic to the random graph R.

In particular, R admits cyclic automorphisms (since the infinite cyclic group satisfies the condition).

A consequence

Corollary: There is no countable abelian B-group.

For let X be a countable abelian group, and $X_2 = \{x \in X : x^2 = 1\}$. Then any translate of a square-root set is a coset of X_2. So if X_2 has infinite index in X, then X satisfies the more general form of Higman's condition.

But if X_2 has finite index in X, then X has finite exponent, and so can be written as $X = Y \times Z$ where Y and Z are infinite; then X is a regular subgroup of the primitive group $S_\infty \wr S_2$ (in the product action).

An example of a group for which it is not known whether it is a B-group is

$$\langle x, y : y^4 = 1, y^{-1}xy = x^{-1} \rangle.$$

The random graph

Take a countable set X of vertices. Choose a random graph on X as follows: decide independently for each pair of vertices, whether they are joined by an edge or not (probability $\frac{1}{2}$ for each outcome).

Theorem: There is a graph R with the property that, almost surely, the above random graph is isomorphic to R.

Moreover, the graph R is *universal* (it contains every finite or countable graph as an induced subgraph) and *homogeneous* (every isomorphism between finite subgraphs of R can be extended to an automorphism of R).

Clearly R is unique up to isomorphism.

The group G of the earlier slide is the automorphism group of R.

Properties of $\mathrm{Aut}(R)$

- $\mathrm{Aut}(R)$ contains a copy of every countable group. (Indeed, if X is countable, then $X \times C_\infty$ satisfies the weaker form of Higman's condition.)
- $\mathrm{Aut}(R)$ is simple: indeed, for any $g, h \in \mathrm{Aut}(R)$ with $g \neq 1$, h is a product of three conjugates of $g^{\pm 1}$. (Truss)
- Every subgroup of index less than $2^{\aleph_0}$ in $\mathrm{Aut}(R)$ contains the pointwise stabiliser of a finite set (and is contained in the setwise stabiliser). This is the so-called *small index property* of R (Hodges *et al.*)

Topology

There is a natural topology on the symmetric group of countable degree, the topology of pointwise convergence. If the domain is $\mathbb{N}$, then $g_n \to g$ if $m^{g_n} = m^g$ for all $n \geq n_0(m)$.

Alternatively, a basis of open neighbourhoods of the identity consists of the stabilisers of all finite tuples of points.

The topology arises from a complete metric on $\mathrm{Sym}(\mathbb{N})$.

A subgroup G of S_n is closed in this topology if and only if G is the automorphism group of some relational structure on its domain Ω (graph, partial order, etc.)

If G is closed and $H \leq G$, then H is dense in G if and only if
G and H have the same orbits on Ω^n for all n.

For example, H is dense in the symmetric group if and only if it is *highly transitive* on Ω.

Topology, continued

The small index property for $\mathrm{Aut}(M)$ is precisely the statement

> Every subgroup of $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ is open.

So if M has the small index property, the topology of $\mathrm{Aut}(M)$ can be recovered from its abstract group structure. Of course, this applies to the random graph R.

In a complete metric space, a subset is *residual* if it contains a countable intersection of open dense sets. Residual sets are non-empty (the *Baire category theorem*), and are regarded as "large".

In common with some other groups like the symmetric group, $\mathrm{Aut}(R)$ contains *generic* elements (having the property that their conjugacy class is residual in the whole group).

Some subgroups of $\mathrm{Aut}(R)$

Here are two remarkable results due to Bhattacharjee and Macpherson.

Theorem: There exist automorphisms f, g of R such that

(a) f has a single cycle on R, which is infinite,

(b) g fixes a vertex v and has two cycles on the remaining vertices (namely, the neighbours and non-neighbours of v),

(c) the group $\langle f, g\rangle$ is free and is transitive on vertices, edges, and non-edges of R, and each of its non-identity elements has only finitely many cycles on R.

Theorem: There is a locally finite group G of automorphisms of R which is dense in $\mathrm{Aut}(R)$ (that is, any isomorphism between finite subgraphs can be extended to an element of G).

Homogeneous structures

A relational structure M on Ω is *homogeneous* if every isomorphism between finite substructures of M can be extended to an automorphism of M.

It is usually simple to decide whether $\mathrm{Aut}(M)$ is primitive or doubly transitive. For example, if M is a graph, then $\mathrm{Aut}(M)$ is primitive if and only if it is not a disjoint union of complete graphs or the complement of one; and it is never doubly transitive unless M is complete or null.

So automorphism groups of homogeneous structures are good places to look for non-B-groups.

It is also interesting to investigate them in their own right, and ask about simplicity, small index property, dense subgroups, etc.

Fraïssé's Theorem

The *age* of a relational structure is the class of isomorphism types of its finite substructures. Fraïssé showed how to recognise the existence of homogeneous structures from their ages.

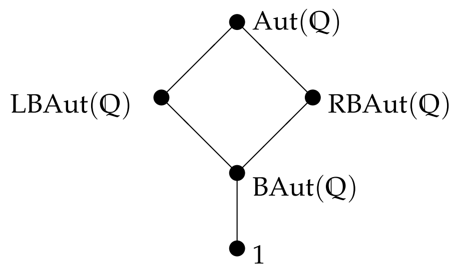
A class $\mathcal{C}$ is the age of a countable homogeneous structure M if and only if $\mathcal{C}$ is closed under isomorphism, closed under taking substructures, contains only countably many structures up to isomorphism, and satisfies the *amalgamation property*. If these conditions hold, then M is unique, and is called the *Fraïssé limit* of $\mathcal{C}$.

For example, the graph R is the Fraïssé limit of the class of all finite graphs (which clearly satisfies the conditions).

Total orders

It is easy to see that $\mathbb{Q}$ is the unique countable homogeneous total order. (This is a consequence of Cantor's Theorem that it is the unique countable dense total order without endpoints. In fact this example was Fraïssé's motivation.)

The normal subgroup structure of $\mathrm{Aut}(\mathbb{Q})$ is known:



Regular subgroups of $\mathrm{Aut}(\mathbb{Q})$

A group X acts regularly on a total order if and only if there is a *right order* on X (a total order invariant under right multiplication). Usually the order will be dense. It seems that "most" right-orderable groups act regularly on $\mathbb{Q}$, and hence are not B-groups.

Problem: Which groups have a right order but not a dense right order?

Right-orderable groups are torsion-free. A free or free abelian group of rank greater than 1 has a dense right-order.

Graphs

Lachlan and Woodrow determined the countable homogeneous graphs. They are the following:

- the disjoint union of complete graphs of the same size, or its complement;
- the Fraïssé limit of the class of graphs containing no complete graph of size m (for fixed $m \geq 3$), or its complement;
- the random graph R.

The first type are not very interesting.

The Fraïssé limit of the class of K_n-free graphs (that is, the unique countable universal homogeneous K_n-free graph) is called the *Henson graph H_n*.

Automorphism groups of Henson's graphs

Here are a few things we don't know about $\mathrm{Aut}(H_n)$ (for $n \geq 3$).

- Is $\mathrm{Aut}(H_n)$ simple?
- Does $\mathrm{Aut}(H_n)$ have the small index property?
- Is it true that $\mathrm{Aut}(H_n)$ and $\mathrm{Aut}(H_m)$ are not isomorphic for $m \neq n$?

  It is known that $\mathrm{Aut}(H_n)$ is not isomorphic to $\mathrm{Aut}(R)$ (using the small index property for $\mathrm{Aut}(R)$).

Henson's graphs as Cayley graphs

A sufficient condition for a countable group to act regularly on H_3 (i.e. to have H_3 as a Cayley graph) is known, Unfortunately it is stronger than the condition for R, so gives us no new non-B-groups.

For $n \geq 4$, we do not have any examples of groups having H_n as a Cayley graph.

Proposition: For $n \geq 4$, H_n is not a *normal* Cayley graph of any group; that is, it is not invariant under both left and right multiplication. In particular, H_n is not a Cayley graph for any abelian group, if $n \geq 4$.

These facts extend a result of Henson, who showed that H_3 admits a cyclic automorphism but H_n does not for $n \geq 4$.

Other structures

In the following cases, the countable homogeneous structures have been determined, but we know very little about their automorphism groups.

- Tournaments (Lachlan): there are just three homogeneous tournaments.

  (A *tournament* is a directed graph in which every pair of vertices is joined by one directed edge.)
- Directed graphs (Cherlin): there are uncountably many.
- Partially ordered sets (Schmerl): there is just one interesting one, the *generic poset*.

Metric spaces

Recall Fraïssé's conditions on $\mathcal{C}$: it should be closed under isomorphism, closed under taking substructures, have only countably many members up to isomorphism, and have the amalgamation property.

Metric spaces can be described as relational structures, with one binary relation for each possible distance. However, they fail the third of Fraïssé's conditions: there are too many 2-point spaces!

However, the class of *rational metric spaces* (with all distances in $\mathbb{Q}$) is a Fraïssé class. If we take its Fraïssé limit, and then take the completion of this, we obtain the Urysohn space $\mathbb{U}$.

The Urysohn space

In a posthumous paper published in 1927, Urysohn showed that there is a Polish space (a complete separable metric space) which is universal (it embeds every Polish space isometrically) and homogeneous (any isometry between finite subsets extends to an isometry of the whole space). Moreover, it is unique up to isometry. This is the *Urysohn space* $\mathbb{U}$.

In fact $\mathbb{U}$ is homeomorphic (though not isometric!) to infinite-dimensional Hilbert space (Uspensky).

Now we can ask many of the same questions about $\mathrm{Aut}(\mathbb{U})$ that we considered for countable homogeneous structures.

Isometries of $\mathrm{Aut}(\mathbb{U})$

- Is it simple as topological group?

  The bounded isometries form a proper normal subgroup. Are the group of bounded isometries and its quotient simple as abstract groups?

- What groups can act with regular dense orbits?

  All that is known is that $\mathbb{Z}$ and the countable elementary abelian 2-group do (so that $\mathbb{U}$ has many abelian group structures), and that the countable elementary abelian 3-group does not.

  Problem: What can be said about the closure of a cyclic group of isometries with dense orbits?

- Other interesting subgroups?

  There is a dense free subgroup, and a dense locally finite subgroup.