Optimal designs and root systems

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The incidence matrix N of the block design is the $v \times b$ matrix with (p, b) entry 1 if $p \in B$, 0 otherwise. The matrix $\Lambda = NN^\top$ is the concurrence matrix, with (p, q) entry equal to the number of blocks containing p and q. It is symmetric, with row and column sums rk, and diagonal entries r.

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A 2-design is optimal in all three senses. But what if no 2-design exists for the given v, k, r?

The question

For a 2-design, the concurrence matrix is $\Lambda = (r - \lambda)I + \lambda J$, where J is the all-1 matrix. Ching-Shui Cheng suggested looking for designs where Λ is a small perturbation of this, say $\Lambda = (r - t)I + tJ - A$, where A is a matrix with small entries (say 0, $+1$, -1). For E-optimality, we want A to have smallest eigenvalue as large as possible (say greater than -2).

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Call such a matrix admissible.

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So we try to determine the admissible matrices by looking for subsets of the root systems.

The case A_n

The vectors of A_n are of the form $e_i - e_j$ for $1 \leq i, j \leq n+1, i \neq j$, where $e_1, \ldots, e_{n+1}$ form a basis for $\mathbb{R}^{n+1}$.

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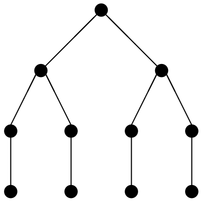
Here is an example (edges directed upwards).



The case D_n

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This case is a bit more complicated. An admissible matrix is represented by a unicyclic graph, whose edges are either directed (if of form $e_i - e_j$) or undirected and signed (if of the form $\pm(e_i + e_j)$). A similar condition for constant row sum can be formulated.

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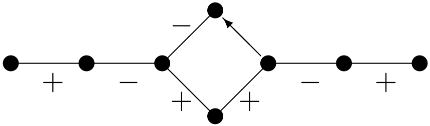

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Here is an example in E_8:

$$\begin{pmatrix} 0 & - & + & + & - & - & + & - \\ - & 0 & - & - & + & + & - & + \\ + & - & 0 & + & - & - & 0 & 0 \\ + & - & + & 0 & - & - & 0 & 0 \\ - & + & - & - & 0 & + & 0 & 0 \\ - & + & - & - & + & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 & - \\ - & + & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix}$$

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An example in E_6 has point set $\{1, 2, 3, 4, 5, 6\}$ and blocks

$$\{123, 125, 125, 134, 136, 136, 146, 156, 234, 245,\\ 246, 246, 256, 345, 345, 356\}.$$

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The next step would be to go on and decide whether any E-optimal block designs are obtained in this way. This has not yet been done!