Cores, hulls and synchronization

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Notation

In this talk, Γ is a graph, G is a group.

For a graph Γ, we use $\omega(\Gamma)$ for the clique number, $\chi(\Gamma)$ for the chromatic number, $\overline{\Gamma}$ for the complement, $\alpha(\Gamma)$ for the independence number (so that $\alpha(\Gamma) = \omega(\overline{\Gamma})$), and $\mathrm{Aut}(\Gamma)$ for the automorphism group of Γ.

Graph homomorphisms

A *homomorphism* from a graph Γ to a graph Γ' is a map from vertices of Γ to vertices of Γ' which maps edges to edges. (We don't care what it does to non-edges.)

Write $\Gamma \to \Gamma'$ if there is a homomorphism, and $\Gamma \equiv \Gamma'$ if there are homomorphisms in both directions.

We use $\mathrm{End}(\Gamma)$ for the semigroup of endomorphisms of Γ (homomorphisms from Γ to Γ).

Example:

- $K_m \to \Gamma$ if and only if $\omega(\Gamma) \geq m$;
- $\Gamma \to K_m$ if and only if $\chi(\Gamma) \leq m$.

Cores

The *core* of Γ is the (unique) smallest graph Δ such that $\Delta \equiv \Gamma$. It is an induced subgraph (indeed, a retract) of Γ.

Thus, the core of Γ is complete if and only if $\omega(\Gamma) = \chi(\Gamma)$.

Proposition 1. *If Γ is vertex-transitive, then so is* $\mathrm{core}(\Gamma)$*. Similarly for other kinds of transitivity.*

Rank 3 graphs

A graph Γ is a *rank* 3 *graph* if its automorphism group is transitive on vertices, ordered edges and ordered non-edges; in other words, $\mathrm{Aut}(\Gamma)$ is a rank 3 permutation group. (The *rank* of a permutation group G on a set V is the number of G-orbits on $V \times V$.)

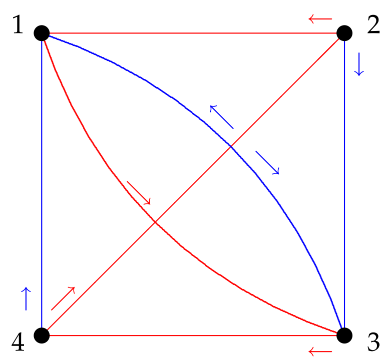
After working out a lot of examples, Cristy Kazanidis and I made the following conjecture:

Conjecture 2. *If Γ is a rank 3 graph, then either the core of Γ is complete, or Γ is a core.*

This is true; the proof came from an unexpected direction: *automata theory*.

The cave

You are in a dungeon consisting of a number of rooms. Passages are marked with coloured arrows. Each room contains a special door; in one room, the door leads to freedom, but in all the others, to instant death. You have a schematic map of the dungeon, but you do not know where you are.



You can check that (Blue, Red, Blue, Blue) is a reset word which takes you to room 3 no matter where you start.

Automata and reset words

An *automaton* is an edge-coloured digraph with one edge of each colour out of each vertex. Vertices are *states*, colours are *transitions*. A *reset word* is a word in the colours such that following edges of these colours from any starting vertex always brings you to the same state. An automaton which possesses a reset word is called *synchronizing*.

Not every finite automaton has a reset word; the *Černý conjecture*, states that, if a reset word exists, then there is one of length at most $(n-1)^2$, where n is the number of states (or rooms in our example).

Synchronizing permutation groups

J. Araújo and B. Steinberg proposed a new approach to the Černý conjecture.

A permutation group G on a set V is *synchronizing* if, given any function $f: V \to V$ which is not a permutation, the semigroup generated by G and f contains a constant function.

Theorem 3. *A permutation group G on V is non-synchronizing if and only if there is a non-complete and non-null graph Γ on V with $\mathrm{core}(\Gamma)$ complete such that $G \le \mathrm{Aut}(\Gamma)$.*

Proof. Let S be a semigroup containing G but no constant function: join v to w if no $f \in S$ satisfies $v^f = w^f$. □

Cores revisited

This gave me the clue for proving the following theorem:

Theorem 4. *Let Γ be a nonedge-transitive graph. Then either*

- $\mathrm{core}(\Gamma)$ *is complete, or*
- Γ *is a core.*

The hull of a graph

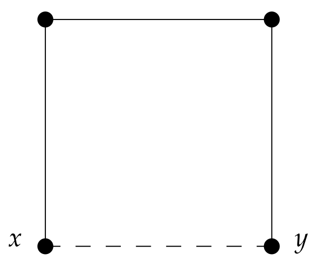
The *hull* of a graph Γ is defined as follows:

- $\mathrm{hull}(\Gamma)$ has the same vertex set as Γ;
- $v \sim w$ in $\mathrm{hull}(\Gamma)$ if and only if there is no element $f \in \mathrm{End}(\Gamma)$ with $v^f = w^f$.

Theorem 5.

- *Γ is a spanning subgraph of $\mathrm{hull}(\Gamma)$;*
- *$\mathrm{End}(\Gamma) \le \mathrm{End}(\mathrm{hull}(\Gamma))$ and $\mathrm{Aut}(\Gamma) \le \mathrm{Aut}(\mathrm{hull}(\Gamma))$;*
- *if $\mathrm{core}(\Gamma)$ has m vertices then $\mathrm{core}(\mathrm{hull}(\Gamma))$ is the complete graph on m vertices.*

An example



No homomorphism can identify x and y, so they are joined in the hull.

Note the increase in symmetry: $|\mathrm{Aut}(\Gamma)| = 2$ but $|\mathrm{Aut}(\mathrm{hull}(\Gamma))| = 8$.

Proof of the theorem

Let Γ be non-edge transitive. Then $\mathrm{hull}(\Gamma)$ consists of Γ with some orbits on non-edges changed to edges. So there are two possibilities:

- $\mathrm{hull}(\Gamma) = \Gamma$. Then $\mathrm{core}(\Gamma) = \mathrm{core}(\mathrm{hull}(\Gamma))$ is complete;
- $\mathrm{hull}(\Gamma)$ is the complete graph on the vertex set of Γ. Then $\mathrm{core}(\Gamma)$ has as many vertices as Γ, so that $\mathrm{core}(\Gamma) = \Gamma$.

Questions about hulls

Let $h(\Gamma)$ be the smallest number of vertices of a graph containing Γ as induced subgraph which is a hull.

Theorem 6. $h(\Gamma) \in \{\chi(\Gamma) - \omega(\Gamma), \chi(\Gamma) - \omega(\Gamma) + 1\}$.

What is the complexity of deciding:

- Is Γ a hull?
- Is $h(\Gamma) = \chi(\Gamma) - \omega(\Gamma)$?
- Is Γ a hull, *given* that $\chi(\Gamma) = \omega(\Gamma)$?

If the third question is hard, so are the other two.

Separating permutation groups

Neumann's separation lemma states:

Proposition 7. *Let G be a transitive permutation group on V, with $|V| = n$, and let A, B be subsets of V. If $|A| \cdot |B| < n$, then there exists $g \in G$ with $A^g \cap B = \emptyset$.*

We call a transitive permutation group *separating* if, for any sets A, B with $|A|, |B| > 1$ and $|A| \cdot |B| = n$, there exists g with $A^g \cap B = \emptyset$.

Separating and synchronizing groups

Proposition 8. 2*-transitive* ⇒ *separating* ⇒ *synchronizing* ⇒ *primitive.*

None of these implications reverses. (But I have only a single example of a permutation group which is synchronizing but not separating, namely $P\Omega(5, 3)$, acting on 40 points.)

Proposition 9.

- *The permutation group G is non-synchronizing if and only if there is a graph Γ (not complete or null) with $\omega(\Gamma) = \chi(\Gamma)$ and $G \leq \mathrm{Aut}(\Gamma)$.*
- *The transitive permutation group G is non-separating if and only if there is a graph Γ (not complete or null) with $\omega(\Gamma) \cdot \alpha(\Gamma) = |V(\Gamma)|$ and $G \leq \mathrm{Aut}(\Gamma)$.*

2-closure

The classes of synchronizing and separating group are upward-closed. They have some downward closure properties too.

The 2*-closure* of a permutation group G on V consists of all the permutations of V which preserve every G-orbit on $V \times V$.

Proposition 10. *A permutation group is synchronizing (resp. separating) if and only if its* 2*-closure is synchronizing (resp. separating).*

This is because failure of these properties is "detected" by a graph admitting the group (and hence admitting its 2-closure).

More general closure properties

This is based on an idea of Arnold and Steinberg.

Let F be a field, and G a permutation group on V. The *F-closure* of G consists of all permutations of V which preserve all the FG-submodules of the permutation module FV.

It is easy to see that $\mathbb{C}$-closure is equivalent to 2-closure.

Proposition 11. *For any field F, a permutation group is synchronizing (resp. separating) if and only if its F-closure is synchronizing (resp. separating).*

An example

The group $\mathrm{PSL}(2, 2^n)$ has permutation actions of degrees $2^{n-1}(2^n \pm 1)$, on the cosets of its maximal dihedral subgroups of orders $2(2^n \mp 1)$. It is 2-closed in both actions.

Suppose that $2^n - 1$ is a Mersenne prime.

The permutation character of the action of degree $2^{n-1}(2^n - 1)$ is the sum of the trivial character and a family of algebraically conjugate characters, whose sum is $\mathbb{Q}$-irreducible. So the $\mathbb{Q}$-closure is the symmetric group, which is trivially separating; so the original group is separating, and hence synchronizing. (This was the example of Arnold and Steinberg.)

The permutation character of the action of degree $2^{n-1}(2^n + 1)$ is equal to the above character plus an irreducible of degree 2^n. So its $\mathbb{Q}$-closure is the group S_{2^n+1} acting on 2-sets, which is separating. (The only invariant graphs are the line graph of K_{2^n+1} and its complement; and if $\Gamma = L(K_{2^n+1})$, then $\omega(\Gamma) = 2^n$, but $\alpha(\Gamma) = 2^{n-1}$.) So again, the original group is separating, and hence synchronizing.