Counting orbits on colourings and flows

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Chromatic polynomial

The chromatic polynomial $P_\Gamma(x)$ of the graph Γ has the property that, for positive integers k, $P_\Gamma(k)$ is the number of proper k-colourings of Γ.

A positive integer k is a root of $P_\Gamma(x)$ if and only if k is smaller than the chromatic number of Γ.

A *chromatic root* is a root of a chromatic polynomial.

- There are no chromatic roots in the intervals $(-\infty, 0)$, $(0,1)$, or $(1, \frac{32}{27}]$ (Jackson)
- Real chromatic roots are dense in $[\frac{32}{27}, \infty)$ (Thomassen)
- Complex chromatic roots are dense in $\mathbb{C}$ (Sokal)

Orbit-counting Lemma

Let G be a group of automorphisms of a graph Γ. We want to count G-orbits on the set of proper colourings of Γ. The key tool is the *Orbit-counting Lemma*:

Theorem 1. *Let G act on a set X. Then the number of orbits of G on X is equal to the average number of fixed points on X of the elements of G:*

$$\#\mathrm{Orbits}(G,X) = \frac{1}{|G|}\sum_{g\in G} \mathrm{fix}_X(g).$$

Said otherwise, the number of orbits is the expected number of fixed points of a random element of G.

Orbital chromatic polynomial

Let g be an automorphism of a graph Γ. Denote by Γ/g the graph obtained by shrinking every cycle of g to a single vertex. The number of k-colourings of Γ fixed by g is equal to the number of colourings of Γ/g. For a colouring is fixed by g if and only if it is constant on the cycles of g (and so induces a proper colouring of Γ/g).

So, if G is a group of automorphisms of Γ, define the *orbital chromatic polynomial* of Γ and G to be

$$OP_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x).$$

Then for positive integers k, the number of orbits of G on the k-colourings of Γ is $OP_{\Gamma,G}(k)$.

Orbital chromatic roots

An *orbital chromatic root* is a root of the orbital chromatic polynomial $OP_{\Gamma,G}$ for some graph Γ and group G.

Taking G to be the trivial group, we see that every chromatic root is an orbital chromatic root.

Here is an example to show that, unlike chromatic roots, orbital chromatic roots can be negative.

Take Γ to be the null graph on n vertices and G the symmetric group S_n. Then an orbit on k-colourings is a choice of n things from a set of k, where repetitions are allowed and order is not significant. This number is $\binom{n+k-1}{n}$. So

$$OP_{\Gamma,G}(x) = \frac{1}{n!}x(x+1)\cdots(x+n-1),$$

with roots $0, -1, \ldots, -(n-1)$.

Orbital chromatic roots, continued

Theorem 2. *Real orbital chromatic roots are dense in $\mathbb{R}$.*

Proof. Take Γ to consist of m triangles and G to be the symmetric group S_m. Then

$$OP_{\Gamma,G}(x) = \frac{1}{m!}q(x)(q(x)+1)\cdots(q(x)+m-1),$$

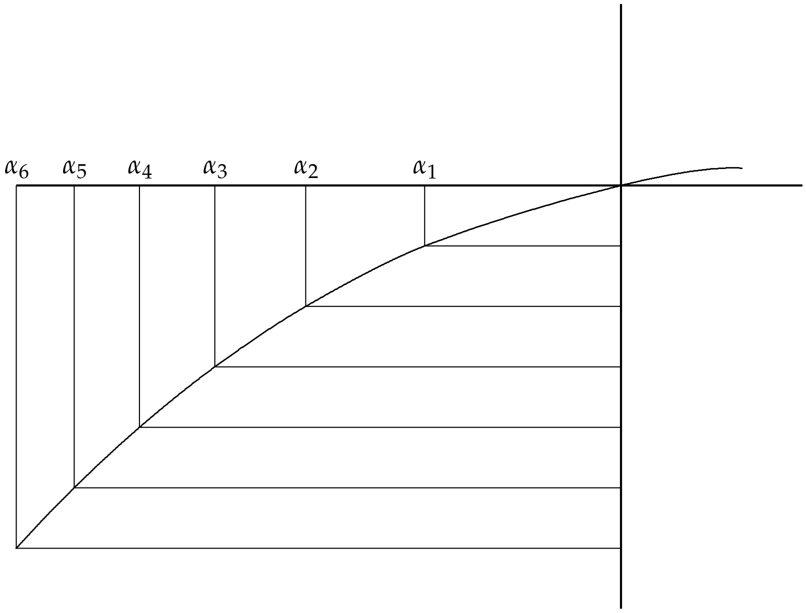
where $q(x) = x(x-1)(x-2)$.

So every root of the equation $x(x-1)(x-2) = -k$ for $k \in \mathbb{N}$ is an orbital chromatic root.

The equation $x(x-1)(x-2) = -k$ has a unique negative root α_k, and the spacing of the α_k becomes denser as k increases.

Now by taking the join with a complete graph of size s, we can translate the roots to the right by any integer s. $\square$

Orbital chromatic roots, continued



Parity, 1

Theorem 3. *The sign of $P_\Gamma(x)$ is $(-1)^v$ for $x < 0$, $(-1)^{v+c}$ for $x \in (0,1)$, and $(-1)^{v+c+b}$ for $x \in (1, \frac{32}{27}]$, where v, c, b denote the numbers of vertices, connected components, and blocks of Γ.*

Since a permutation is even if and only if the numbers of points and cycles are congruent mod 2, we have:

Theorem 4. *Let V and C be the sets of vertices and connected components of Γ.*

- *(a) Suppose that every element of G is an even permutation of V. Then $OP_{\Gamma,G}(x)$ has no roots in $(-\infty, 0)$.*
- *(b) Suppose that every element of G is an even permutation of $V \cup C$. Then $OP_{\Gamma,G}(x)$ has no roots in $(0,1)$.*

Parity, 2

The analogous result for blocks is false. The reason is that, while vertices and connected components of Γ/g correspond to cycles of g on vertices and connected components of Γ, the analogous statement is not true for blocks.

Problem 5. *Is it true that the real roots of* $OP_{\Gamma,G}(x)$ *are bounded above by the largest real root of* $P_\Gamma(x)$*?*

This is known in some cases, for example when Γ is a null graph and G any permutation group on the vertices: in this case $OP_{\Gamma,G}(x)$ is a specialisation of the cycle index of G, obtained by putting all the variables equal to x.

Problem 6. *Is it true that the real roots of* $OP_{\Gamma,G}(x)$*, where* Γ *is* 2*-connected and* G *consists of even permutations of the vertex set, are dense in* $[1,\infty)$*?*

Under these hypotheses, the only root less than 1 is 0.

Flow and tension polynomials

Take a fixed but arbitrary orientation of the edges of Γ. A *flow* on Γ with values in the abelian group A is a function from the oriented edges of Γ to A with the property that the net flow into any vertex is zero (calculated in A).

The number of nowhere-zero A-flows is a polynomial in A, the *flow polynomial*, and doesn't depend on the structure of A.

Dually, a *tension* is a function from oriented edges to A so that the net flow around any circuit is zero. The number of nowhere-zero tensions is also a polynomial in $|A|$.

The flow and tension polynomials are specialisations of the *Tutte polynomial* of Γ.

Orbital flow polynomial

Things are more complicated for orbits on flows: the answer does depend on the structure of A. Precisely, given a group G of automorphisms of the graph Γ, there is a polynomial $OF_{\Gamma,G}(x_0, x_1, x_2, ldots)$ in indeterminates indexed by $\mathbb{N}$, such that the number of G-orbits on nowhere-zero A-flows on Γ is $OF_{\Gamma,G}(a_0, a_1, \ldots)$, where a_i is the number of solutions of $ix = 0$ in the abelian group A. Note that $a_0 = |A|$ and $a_1 = 1$.

This orbital flow polynomial is a specialisation of an *orbital Tutte polynomial* of Γ and G, which involves two potentially infinite families of variables.

If the variable x_i actually occurs, then G must contain an element of order i. Thus we recover Tutte's observation by taking G to be the trivial group.

Orbital flow roots

An *orbital flow root* is a root of the polynomial $OF^1_{\Gamma,G}$ obtained by putting $x_i = 1$ for all $i > 0$, for some pair (Γ, G). It counts orbits on flows in the case where $\gcd(|A|, |G|) = 1$.

Theorem 7. *Real orbital flow roots are dense in* $(-\infty, 0)$.

The graphs used to prove this theorem can be taken to be connected simple planar graphs.

The method of proof also shows that every value $1/k$, for $k \in \mathbb{N}$, is a limit point of real orbital flow roots.

The limitation in the method is that there is no way to "translate" orbital flow roots to the right, as there is for orbital chromatic roots!

Invariant factors and duality, 1

Let R be a principal ideal domain. Given an $m \times n$ matrix M over R, we define the *row space* of $\rho(M)$ and the *null space* $\nu(M)$ as usual:

$$\begin{aligned}\rho(M) &= \{yM : y \in R^m\},\\ \nu(M) &= \{x \in R^n : Mx^\top = 0\}.\end{aligned}$$

M can be put into *Smith normal form* by elementary row and column operations: this is a matrix with r non-zero diagonal elements $d_1, \ldots, d_r$ and all other entries zero, where d_i divides d_{i+1} for $i = 1, \ldots, r-1$. The elements $d_1, \ldots, d_r$ are uniquely determined up to multiplication by units of R. They are the *invariant factors* of M. By convention, we also take 0 to be an invariant factor with multiplicity $n - r$, so that there are n invariant factors in all.

Invariant factors and duality, 2

Two matrices M and M^* over the PID R are *dual* if the row space of M is equal to the null space of M^* and *vice versa*.

A matrix is *totally unimodular* if every subdeterminant is zero or a unit. (This property is *not* preserved by elementary operations.)

Theorem 8. *Let M be a matrix over R. Then the following are equivalent:*

- *M has a dual;*
- *all invariant factors of M are zero or units;*
- *M is equivalent (by elementary row and column operations) to a totally unimodular matrix.*

If Γ is a graph with oriented edges, and M and M^* are its signed vertex-edge and cycle-edge incidence matrices, then M and M^* are dual.

Orbital Tutte polynomial, 1

Assume that (M, M^*) is a dual pair over a principal ideal domain R. The linearly independent sets of columns of M are the independent sets of a *matroid*. The linearly independent sets of columns of M^* form the *dual matroid*.

An *automorphism* of M to be an automorphism of the free module R^n (where n is the number of columns of M) which preserves the row space and null space of M.

If g is an automorphism of M (represented as an $n \times n$ matrix), and 1 is the identity matrix, set

$$M_g = \begin{pmatrix} M \\ g-1 \end{pmatrix}, \qquad M_g^* = \begin{pmatrix} M^* \\ g-1 \end{pmatrix}.$$

For any subset S of $E = \{1, \ldots, n\}$, and any matrix N with n columns, we let $N[S]$ be the submatrix of N consisting of the columns with indices in S.

Orbital Tutte polynomial, 2

Take two sets $(x_i : i \in I)$ and $(x_i^* : i \in I)$ of indeterminates, where the index set I is the set of associate classes in R. For any matrix N, let $x(N)$ be the monomial defined as follows: take the invariant factors of N (completed with zeros so that the number of them is equal to the number of columns of N), and multiply the corresponding indeterminates. Define $x^*(N)$ similarly, using the other set of indeterminates.

Now let G be a finite group of automorphisms of M, and define the *orbital Tutte polynomial* $OT(M, G)$ in the indeterminates $(x_i, x_i^* : i \in I)$ as follows:

$$OT(M, G) = \frac{1}{|G|} \sum_{g \in G} \sum_{S \subseteq E} x(M_g[S]) x^*(M_g^*[E \setminus S]).$$

Theorem 9. *If G is the trivial group, then $OT(M, G)$ involves only x_0, x_1, x_0^* and x_1^*; the substitution $x_1 = x_1^* = 1$, $x_0 = y - 1$, $x_0^* = x - 1$ gives the Tutte polynomial of M.*

Graphs

For a graph Γ, let $OT(\Gamma, G)$ denote $OT(M, G)$, where M is the signed vertex-edge incidence matrix of Γ.

Theorem 10. *Let M be the incidence matrix of a graph Γ over $\mathbb{Z}$, and let G be a group of automorphisms of Γ. Let A be a finite Abelian group. Then the substitution $x_i \leftarrow \alpha_i(A)$, $x_i^* \leftarrow -1$ (for all i) in $OT(\Gamma, G)$ gives the number of G-orbits on nowhere-zero A-flows on Γ, while the substitution $x_i \leftarrow -1$, $x_i^* \leftarrow \alpha_i(A)$ gives the number of G-orbits on nowhere-zero A-tensions on Γ.*

Theorem 11. *Let Γ be a connected graph. Then the orbital chromatic polynomial of $(\Gamma, G; k)$ is obtained from the orbital tension polynomial by substituting $x_0^* = k$, $x_i^* = 1$ for $i > 0$, and multiplying by k.*

Supports

Theorem 12.

- *Let f_m be the number of G-orbits on A-flows supported on precisely $n - m$ edges of Γ. Then*

  $$\sum f_m x^m = OT(\Gamma, G; x_i \leftarrow \alpha_i(A), x_i^* \leftarrow x - 1).$$

- *Let t_m be the number of G-orbits on A-tensions supported on precisely $n - m$ edges of Γ. Then*

  $$\sum t_m x^m = OT(\Gamma, G; x_i \leftarrow x - 1, x_i^* \leftarrow \alpha_i(A)).$$

Orbital weight enumerator

A linear code over $\mathrm{GF}(q)$ is the row space of a *generator matrix* M over $\mathrm{GF}(q)$, and is the null space of a *parity check matrix* M^* (the generator matrix of the dual code). These matrices are duals; so if G is a group of automorpisms of C, the orbital Tutte polynomial P is defined as before. Since $\mathrm{GF}(q)$ is a field, P involves only the variables x_0, x_1, x_0^*, x_1^*.

The *orbital weight enumerator* is the homogeneous polynomial

$$W_{C,G}(X, Y) = \sum_{i=0}^{n} a_i X^{n-i} Y^i,$$

where a_i is the number of G-orbits on words of weight i in C.

Theorem 13. *Let G be an automorphism group of a linear code over* $\mathrm{GF}(q)$*. Then the orbital weight enumerator C is obtained from the orbital Tutte polynomial by the substitution*

$$x_0 = x_1 = X - Y, \quad x_0^* = qY, \quad x_1^* = Y.$$

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