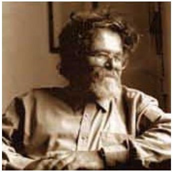
Posets, homomorphisms, and homogeneity

Peter J. Cameron

Queen Mary University of London

p.j.cameron@qmul.ac.uk



Dedicated to Jarik Nešetřil on his sixtieth birthday

HAPPY BIRTHDAY JARIK!

Summary

Jarik Nešetřil has made deep contributions to all three topics in the title, and we began thinking about connections between them when I spent six weeks in Prague in 2004. In this talk I want to survey the three topics and their connections. I will be reporting a theorem by my student Debbie Lockett.

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- Construction of the generic poset
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Universality and homogeneity

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- homogeneous if every isomorphism between finite substructures of M can be extended to an automorphism of M (an isomorphism $M \to M$).

The age of a relational structure M is the class $\mathcal{C}$ of all finite structures embeddable in M.

Fraïssé's Theorem

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In about 1950, Fraïssé gave a necessary and sufficient condition on a class $\mathcal{C}$ of finite structures for it to be the age of a countable homogeneous structure M.

The key part of this condition is the *amalgamation property*: two structures in $\mathcal{C}$ with isomorphic substructures can be "glued together" so that the substructures are identified, inside a larger structure in $\mathcal{C}$.

Moreover, if $\mathcal{C}$ satisfies Fraïssé's conditions, then M is unique up to isomorphism; we call it the Fraïssé limit of $\mathcal{C}$.

Ramsey theory

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This gives a programme for determining the Ramsey classes: first find classes satisfying the amalgamation property, and then decide whether they have the Ramsey property.
The converse is false in general, but Jarik Nešetřil recently showed that the class of finite metric spaces is a Ramsey class.

The random graph

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- R is the unique countable universal homogeneous graph;
- R is the countable random graph; that is, if edges of a countable graph are chosen independently with probability $\frac{1}{2}$, then the resulting graph is isomorphic to R with probability 1 (Erdős and Rényi);
- R is the generic countable graph (this is an analogue of the Erdős–Rényi theorem, with Baire category replacing measure).

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We do not need all of ZF for this; in particular, Choice is not required. The crucial axiom turns out to be Foundation.

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There is no known direct construction of $\mathbb{P}$ similar to the constructions of R. I now outline a nice recursive construction by Hubička and Nešetřil.

Set theory with an atom

Take a countable model of set theory with a single atom $\Diamond$.
Now let M be any set not containing $\Diamond$. Put

$$M_L = \{A \in M : \Diamond \notin A\},$$
$$M_R = \{B \setminus \{\Diamond\} : \Diamond \in B \in M\}.$$

Then neither M_L nor M_R contains $\Diamond$.

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In the other direction, given two sets P, Q whose elements don't contain $\diamondsuit$, let $(P \mid Q) = P \cup \{B \cup \{\diamondsuit\} : B \in Q\}$. Then $(P \mid Q)$ doesn't contain $\diamondsuit$.
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Note that any set not containing $\diamondsuit$ can be represented in terms of sets not involving $\diamondsuit$ by means of the operation $(. \mid .)$
For example, $\{\emptyset, \{\diamondsuit\}\}$ is $(\{\emptyset\} \mid \{\emptyset\})$.

The generic poset

Let $\mathcal{P}$ be the collection of the sets M not containing $\diamondsuit$ defined by the following recursive properties:
Correctness: $M_L \cup M_R \subseteq \mathcal{P}$ and $M_L \cap M_R = \emptyset$;
Ordering: For all $A \in M_L$ and $B \in M_R$, we have

$$(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset.$$

Completeness: $A_L \subseteq M_L$ for all $A \in M_L$, and $B_R \subseteq M_R$ for all $B \in M_R$.

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Theorem

The above-defined structure is isomorphic to the generic poset $\mathbb{P}$.

Homomorphisms

A homomorphism $f : M \to N$ between relational structures of the same type is a map which preserves the relations. For example, if M and N are posets with the strict order relation $<$, then a f is a homomorphism if and only if

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For most of this talk I will consider the strict order.

Notions of homogeneity

We say that a relational structure X has property HH if every homomorphism between finite substructures of X can be extended to a homomorphism of X. Similarly, X has property MH if every monomorphism between finite substructures extends to a homomorphism. There are six properties of this kind that can be considered: HH, MH, IH, MM, IM, and II. (It is not reasonable to extend a map to one satisfying a stronger condition!) Note that II *is equivalent to the standard notion of homogeneity defined earlier*.

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These properties are related as follows (strongest at the top):

$$\begin{tikzcd}
\mathrm{II} \arrow[dr] & & \mathrm{MM} \arrow[dl] \arrow[dr] & & \mathrm{HH} \arrow[dl] \\
& \mathrm{IM} \arrow[dr] & & \mathrm{MH} \arrow[dl] & \\
& & \mathrm{IH} & &
\end{tikzcd}$$

Extensions of $\mathbb{P}$

We can recognise $\mathbb{P}$ by the property that, if A, B and C are pairwise disjoint finite subsets with the properties that $A < B$, no element of A is above an element of C, and no element of B is below an element of C, then there exists a point z which is above A, below B, and incomparable with C.

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Extensions of $\mathbb{P}$ (posets X with the same point set, in which $x < y$ in $\mathbb{P}$ implies $x < y$ in X) can be recognised by a similar property: if A and B are finite disjoint sets with $A < B$, then there exists a point z satisfying $A < z < B$.

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Using this, it can be shown that any extension of $\mathbb{P}$ has the properties MM and HH (and hence all the earlier properties except II).

Properties

If an IH poset P is not an antichain, then it has the following property:

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This is easy to see in the case $Q = \emptyset$ (so that P has no least or greatest element). In general, suppose that $Q < z$, and $z < z'$. Extend the isomorphism fixing Q and mapping z' to z; if z'' is the image of z, then $Q < z'' < z$.

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Taking Q to be a singleton, we see that P is dense.

X-free posets

We say that a countable poset is X-free if it satisfies the following:

> *If A and B are 2-element antichains with $A < B$, then there does not exist a point z with $A < z < B$.*

Such a point z together with A and B would form the poset X.

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Take a discrete tree T; for each pair (x, y) in T such that y covers x, add a copy of the open rational interval $(0, 1)$ between x and y; and delete the points of T. This poset is vacuously X-free, and also has the property that for any finite Q, $\{z : z < Q\}$ has no maximal element and $\{z : z > Q\}$ has no minimal element.

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Any poset with these two properties can be shown to be HH and MM. This gives $2^{\aleph_0}$ non-isomorphic HH and MM posets.

Lockett's Theorem

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- *A countable poset P has one of these properties if and only if one of the following holds:*
  - *P is an antichain;*
  - *P is the union of incomparable copies of $\mathbb{Q}$;*
  - *P is an extension of the generic poset $\mathbb{P}$;*
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Thus, for posets, the earlier diagram simplifies:

$$\begin{array}{c} \text{II} \\ \downarrow \\ \text{IM} = \text{IH} = \text{MM} = \text{MH} = \text{HH} \end{array}$$

Non-strict orders

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Lockett has shown that the diagram for non-strict partial order is

$$\begin{array}{c} \mathrm{II} \\ \downarrow \\ \mathrm{IM} = \mathrm{MM} \\ \downarrow \\ \mathrm{IH} = \mathrm{MH} = \mathrm{HH} \end{array}$$

Graphs

For graphs, it is not true that the five classes coincide. Jarik and I showed that a countable MH graph either is an extension of the random graph R (containing it as a spanning subgraph), or has bounded claw size. Apart from disjoint unions of complete graphs (containing no $K_{1,2}$), no examples with bounded claw size are known. Extensions of R are MM and HH.

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The homogeneous (II) graphs were all found by Lachlan and Woodrow. They are disjoint unions of complete graphs and their complements; the Fraïssé limit of the class of K_n-free graphs ($n \geq 3$) and its complement; and the random graph. We don't know what happens for IH or IM.