Some aspects of codes over rings

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This is work by two of my students,
Josephine Kusuma and Fatma Al-Kharoosi


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Summary

- Codes over rings and orthogonal arrays

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- $\mathbb{Z}_4$ codes and Gray map images

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- Generalisation to $\mathbb{Z}_{p^n}$

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We define the (Hamming) metric d_H, the inner product of words, and the dual of a code, over a ring R just as for codes over fields.

Orthogonal arrays

A code C over an alphabet R is an *orthogonal array of strength* t if, given any set of t coordinates $i_1, \ldots, i_t$, and any entries $r_1, \ldots, r_t \in R$, there is a constant number of codewords $c \in C$ such that $c_{i_k} = r_k$ for $k = 1, \ldots, t$.

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The strength of a code C is the largest t for which C is an orthogonal array of strength t.

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This was proved by Delsarte for codes over fields. The generalisation is not completely straightforward. It depends on the following property of rings (which, here, mean finite commutative rings with identity).

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Now the theorem is the case $n = 1$ of the coding result: a code of length 1 is just an ideal of R and the dual code is its annihilator. The general case is then proved by a careful induction.
It is not true that $|\mathrm{Ann}(I)| = |R|/|I|$ for any ideal I, and hence not true that $|C^{\perp}| = |R|^n/|C|$ for any code over the ring R.
However this does hold for rings such as the integers mod q for positive integers q, or for finite fields.

The Gray map

The Lee metric d_L on $\mathbb{Z}_4^n$ is defined coordinatewise:

$$d_L(v,w) = \sum_{i=1}^{n} d_L(v_i, w_i),$$

where the Lee metric on $\mathbb{Z}_4$ is given by the rule that $d_L(a,b)$ is the number of steps from a to b when the elements of $\mathbb{Z}_4$ are arranged round a circle.

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The Gray map γ is a non-linear map from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$, which is an isometry from the Lee metric on $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$. It is also defined coordinatewise: on $\mathbb{Z}_4$ we have

$$\gamma(0) = 00, \quad \gamma(1) = 01, \quad \gamma(2) = 11, \quad \gamma(3) = 10.$$

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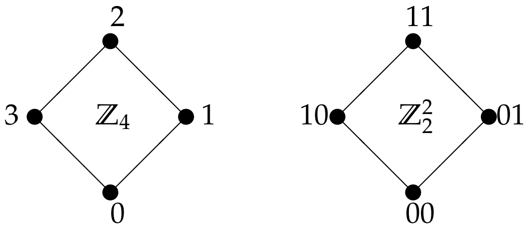
It was introduced by Hammons *et al.* in their classic paper showing that certain nonlinear binary codes such as the Nordstrom–Robinson, Preparata and Kerdock codes are Gray map images of linear $\mathbb{Z}_4$-codes.

The Gray map



A theorem and a conjecture

Conjecture

Let C be a linear code over $\mathbb{Z}_4$ and C' its Gray map image. Then the strength of C' is one less than the minimum Lee weight of $C^{\perp}$.

Note that the strength of C is one less than the minimum Hamming weight of $C^{\perp}$.

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Moreover, if C and C' have strength t and t' respectively, then it is known that $t \leq t' \leq 2t + 1$. (This would follow from the truth of the conjecture.)

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Theorem

Let C be a linear code over $\mathbb{Z}_4$ and C' its Gray map image. Then the strength of C' is at most the minimum Lee weight of $C^\perp$ minus one.

A classification of $\mathbb{Z}_4$-codes

With any $\mathbb{Z}_4$-code C, we can associate a pair (C_1, C_2) of binary codes as follows. (This is a special case of a construction due to Eric Lander).

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Algebraically, there is a homomorphism from C to C_1 with kernel (isomorphic to) C_2; so C is an extension of C_2 by C_1. So you should expect cohomology to come in somewhere …

The class $\mathcal{C}(C_1, C_2)$

We note that $C_1 \leq C_2$. For, given any word $c \in C_1$, let c' be a word in C mapping onto c; then $2c'$ has all entries 0 or 2 and produces the word $c \in C_2$.

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If the length is n, and $\dim(C_i) = k_i$ for $i = 1, 2$, then $|\mathcal{C}(C_1, C_2)| = 2^{k_1(n-k_2)}$.

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Given C_1 and C_2, what can we say about properties of the codes in $\mathcal{C}(C_1, C_2)$?

Generator matrices

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Since Y is $k_1 \times (n - k_2)$, where $k_i = \dim(C_i)$, this gives the formula for $|\mathcal{C}(C_1, C_2)|$.

Weight enumerators

The symmetrized weight enumerator of a $\mathbb{Z}_4$-code C is the three-variable homogeneous polynomial

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Theorem

The average of the symmetrized weight enumerators of the codes in $\mathcal{C}(C_1, C_2)$ *is*

$$\frac{|C_2|}{2^n} \left(W_{C_1}(x+y, 2z) - (x+y)^n \right) + W_{C_2}(x, y).$$

Weight enumerators, continued

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A detailed example is given later.

$\mathcal{C}(C_1, C_2)$ as an affine space

The fact that $|\mathcal{C}(C_1, C_2)|$ is a power of 2 is not a coincidence: the group $C_1^* \otimes (\mathbb{Z}_2^n / C_2)$ acts on this set by translation. (C_1^* is the dual space of C_1.)

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For $C_1^* \otimes \mathbb{Z}_2^n$ acts on $\mathcal{C}$ by the rule

$$(f \otimes w)(c) = c + d(f(c \bmod 2))w$$

where d is the "doubling" map $0 \to 0, 1 \to 2$ from $\mathbb{Z}_2$ to $\mathbb{Z}_4$, and the kernel of the action is $C_1^* \otimes C_2$.

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So if we fix a reference code in $\mathcal{C}$ to act as origin, there is a bijection between $\mathcal{C}$ and $C_1^* \otimes (\mathbb{Z}_2^n / C_2)$.

Another group action

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Multiplying all coordinates by 3 fixes all the codes, so the group $\mathbb{Z}_2^{n-1}$ acts.
These two groups generate their semidirect product $(\mathbb{Z}_2^{n-1}) : (\mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2))$.

First cohomology

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A derivation is a map $d : G \longrightarrow A$ satisfying $d(g_1 g_2) = d(g_1)^{g_2} + d(g_2)$. It is inner if there is an element $a \in A$ such that $d(g) = a^g - a$.

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If A is a vector space and G a linear group, then $A : G$ is a group of affine transformations of A; the stabilizer of the zero vector is a complement, and a complement is conjugate to G if and only if it fixes a vector.

A case study

A very interesting case is that in which $C_1 = C_2$ is the extended Hamming code of length 8. The class $\mathcal{C}(C_1, C_2)$ includes the "octacode" whose Gray map image is the non-linear Nordstrom–Robinson code of length 16.

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The class $\mathcal{C}$ in this case admits the group $G = (\mathbb{Z}_2^7) : \mathrm{AGL}(3, 2)$ (the first factor corresponds to coordinate sign changes, the second is the common automorphism group of C_1 and C_2).

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The class $\mathcal{C}$ in this case admits the group $G = (\mathbb{Z}_2^7) : \mathrm{AGL}(3, 2)$ (the first factor corresponds to coordinate sign changes, the second is the common automorphism group of C_1 and C_2).
The cohomology group $H^1(G, W)$ is non-zero, and indeed the class $\mathcal{C}$ realises an outer derivation.

A case study, continued

The table gives the orbit lengths of G on $\mathcal{C}$, the symmetrized weight enumerator of a code in each orbit, and the number of orbits of the subgroup $\mathrm{AGL}(3,2)$ (the automorphism group of the extended Hamming code). Here

$$F(x,y,z) = x^8 + 14x^4y^4 + y^8 + 16z^8 + 112xyz^4(x^2+y^2)$$

is the weight enumerator of the octacode, and

$$E(x,y,z) = 4z^4(x-y)^4.$$

The data

Orbit	SWE	#perm orbits
7168	F+5E	19
896	F+6E	7
21504	F+4E	24
21504	F+3E	27
3584	F+4E	14
896	F+4E	4
7168	F+2E	8
2688	F+2E	8
128	F	3

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The average SWE is $F + \frac{7}{2}E$, in agreement with Theorem 6.

Problems

- In the example, the symmetrized weight enumerators of the codes in $\mathcal{C}(C_1, C_2)$ lie on a line in the space of polynomials. In general, Fatma's work shows that they always lie on a relatively low-dimensional space. Can one calculate this dimension, in terms of C_1 and C_2?

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- Can one give lower bounds for the number of different SWEs that occur?
- Can one give necessary and sufficient conditions for the element of $H^1(\mathbb{Z}_2^{n-1} : \mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2), C_1^* \otimes (\mathbb{Z}_2^n / C_2))$ to be non-zero?

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- Can one give lower bounds for the number of different SWEs that occur?
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- Can one calculate the number of orbits of $\mathbb{Z}_2^{n-1} : \mathrm{Aut}(C_1) \cap \mathrm{Aut}(C_2)$ on $\mathcal{C}(C_1, C_2)$? (This number is not greater than the number of orbits on $C_1^* \otimes (\mathbb{Z}_2^n / C_2)$, and is equal if the cohomology element is zero.)

More generally ...

Following Eric Lander's method, we can associate a chain of r codes over $\mathbb{Z}_p$ with any code over $\mathbb{Z}_{p^r}$. The ith code consists of words of C with all entries divisible by p^{i-1}, read modulo p^i and then "divided" by p^{i-1} to give a $\mathbb{Z}_p$-code.

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One can ask the inverse question: Given a chain of $\mathbb{Z}_p$-codes, how many $\mathbb{Z}_{p^r}$ codes give rise to this chain, and what can be said about their properties?

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Almost nothing is known about this!

… for your attention!