Symmetry in mathematics and mathematics of symmetry

Peter J. Cameron
Queen Mary
University of London
p.j.cameron@qmul.ac.uk

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Symmetry in mathematics

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I begin with three classical examples, one from geometry, one from model theory, and one from graph theory, to show the contribution of symmetry to mathematics.

Example 1: Projective planes

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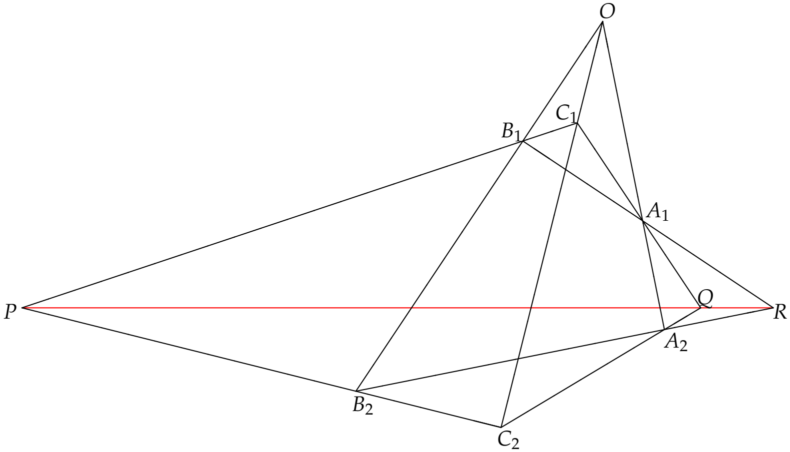
Hilbert showed:

Theorem

A projective plane can be coordinatised by a skew field if and only if it satisfies Desargues' Theorem.

Desargues' Theorem



How not to prove Hilbert's Theorem

Set up coordinates in the projective plane, and define addition and multiplication by geometric constructions.

Then prove that, if Desargues' Theorem is valid, then the coordinatising system satisfies the axioms for a skew field.

This is rather laborious! Even the simplest axioms require multiple applications of Desargues' Theorem.

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Desargues' Theorem is equivalent to the assertion:

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Now the additive group of the coordinatising skew field is the group of central collineations with centre O and axis L where $O \in L$; the multiplicative group is the group of central collineations where $O \notin L$.
So all we have to do is prove the distributive laws (geometrically) and the commutative law of addition (which follows easily from the other axioms).

Example 2: Categorical structures

A first-order language has symbols for variables, constants, relations, functions, connectives and quantifiers. A structure M over such a language consists of a set with given constants, relations, and functions interpreting the symbols in the language. It is a model for a set Σ of sentences if every sentence in Σ is valid in M.

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So there are only two types of categoricity: countable and uncountable.

Oligomorphic permutation groups

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Example

Let G be the group of order-preserving permutations of the set $\mathbb{Q}$ of rational numbers. Two n-tuples $\bar{a}$ and $\bar{b}$ of rationals lie in the same G-orbit if and only if they satisfy the same equality and order relations, that is,

$$a_i = a_j \Leftrightarrow b_i = b_j, \quad a_i < a_j \Leftrightarrow b_i < b_j.$$

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So the number of orbits of G on $\mathbb{Q}^n$ is equal to the number of preorders on an n-set.

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Let M be a countable first-order structure. Then the theory of M is countably categorical if and only if the automorphism group $\mathrm{Aut}(M)$ is oligomorphic.

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Example

Cantor showed that $\mathbb{Q}$ is the unique countable dense linearly ordered set without endpoints. So $\mathbb{Q}$ (as ordered set) is countably categorical.
We saw that $\mathrm{Aut}(\mathbb{Q})$ is oligomorphic.

Oligomorphic groups and counting

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A number of general properties of such sequences are known.

To state the next results, we let G be a permutation group on Ω; let $F_n(G)$ be the number of orbits of G on ordered n-tuples of distinct elements of Ω, and $f_n(G)$ the number of orbits on n-element subsets of Ω.

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Typically, $F_n(G)$ counts labelled combinatorial structures and $f_n(G)$ counts unlabelled structures. Both sequences are non-decreasing.

Sequences from oligomorphic groups

Theorem

There exists an absolute constant c such that, if G is an oligomorphic permutation group on Ω which is primitive (i.e. preserves no non-trivial partition of Ω), then either

- $f_n(G) = 1$ *for all n; or*
- $f_n(G) \geq c^n / p(n)$ *and* $F_n(G) \geq n!\, c^n / q(n)$*, where p and q are polynomials.*

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Theorem

Let G be a group with $f_n(G) = 1$ for all n (in the above notation). Then either

- *G preserves or reverses a linear or circular order on Ω; or*
- $F_n(G) = 1$ *for all n. (In this case we say that G is highly transitive on Ω.)*

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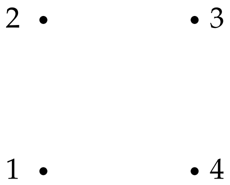

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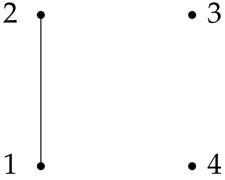

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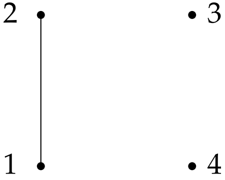

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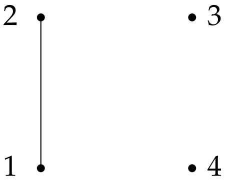

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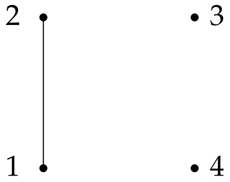

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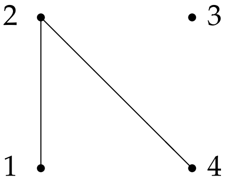

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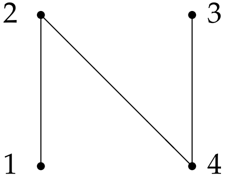

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Finite random graphs

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But . . .

The Erdős–Rényi Theorem

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We will say more about R and its automorphism group later.

Symmetry and groups

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Is every group the symmetry group of something?
This ill-defined question has led to a lot of interesting research.
We have to specify

- whether we consider the group as a permutation group (so the action is given) or as an abstract group;
- what kinds of structures we are considering.

As a permutation group

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Problem

Which permutation groups of countable degree are automorphism groups of relational structures over finite relational languages?

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Here are a couple of open questions.

- Every group is the collineation group of a projective plane. But is every *finite* group the automorphism group of a *finite* projective plane?
- Is every finite group the *outer automorphism group* (automorphisms modulo inner automorphisms) of some finite group?

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Theorem

A finite simple group is one of the following:

- *a cyclic group of prime order;*
- *an alternating group A_n, for $n \geq 5$;*
- *a group of Lie type, roughly speaking a matrix group of specified type over a finite field modulo scalars;*
- *one of the* 26 *sporadic groups, whose orders range from* 7 920 *to* 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000.

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To apply this theorem, we need to understand these simple groups well!

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- Apply CFSG.

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- They are small: order at most $n^{c \log \log n}$ with "known" exceptions.
- They have small base size: almost simple primitive groups have base size bounded by an absolute constant with known exceptions.

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Problem

Find an "elementary" proof!

Related questions

- The FKS theorem doesn't tell us which prime! Does there exist a function $f(p, b)$ such that, if $n = p^a \cdot b$ with $a \geq f(p, b)$, then a transitive permutation group of degree n contains a fixed-point-free element of p-power order?

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The problem in these cases is that there is no simple reduction to primitive groups.

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A structure M is homogeneous if every isomorphism between finite substructures of M can be extended to an automorphism of M; in other words, "any local symmetry is global".

Local or global?

Mathematicians typically consider the second, global, notion, but what about the first, local, notion, and what is the relationship between them?

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Example

The pentagon is homogeneous.

Homogeneous structures

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This is now a very active field bordering logic, group theory, combinatorics, dynamics, etc.

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- If $g, h \in G$ with $g \neq 1$ then h is the product of three conjugates of g.
- Every countable group is embeddable as a semiregular subgroup of G.

Other applications of Fraïssé's method

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Using a variant of Fraïssé's method, Hrushovski and others have constructed various generalised polygons, distance-transitive graphs, etc., with lots of symmetry.

More generally . . .

The condition of homogeneity can be weakened in various ways, using the notion of *homomorphism* or *monomorphism* in place of *isomorphism*. Investigation of these ideas is quite recent. If H='homo', M='mono', and I='iso', we can say that a structure X has the IH-property if any isomorphism between finite substructures of X extends to a homomorphism of X, with similar definitions for MH, HH, IM, and MM (and, indeed, II, which is "classical" homogeneity).

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Here is a sample result due to Debbie Lockett.

Theorem
For countable partially ordered sets with strict order, the classes IH, MH, HH, IM, and MM all coincide, and are strictly weaker than II.