# Association schemes and permutation groups

Peter J Cameron

School of Mathematical Sciences
Queen Mary, University of London
London E1 4NS, U.K.
p.j.cameron@qmw.ac.uk

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## Definitions

$A_0, \dots, A_{r-1}$ are $\Omega \times \Omega$ zero-one matrices. Such matrices represent subsets of $\Omega \times \Omega$.

**Coherent configuration:**

(a) $\sum_{i=0}^{r-1} A_i = J$, the all-$1$ matrix. (The corresponding subsets form a partition of $\Omega \times \Omega$.)

(b) $\sum_{i=0}^{s-1} A_i = I$. (The diagonal is a union of classes.)

(c) $A_i^\top = A_{i^*}$, where $*$ is an involution on $\{0, \dots, r-1\}$.

(d) $A_i A_j = \sum_{k=0}^{r-1} p_{ij}^k A_k$. (The matrices span an algebra.)

**Association scheme:** $*$ is the identity, i.e. all the relations are symmetric. (This implies that $s = 1$, that is, $A_0 = I$: the diagonal is a single class.)



## Permutation groups

Let $G$ be a permutation group on $\Omega$. Then the characteristic functions of the orbits of $G$ on $\Omega \times \Omega$ form a coherent configuration $\mathcal{K}(G)$.

$\mathcal{K}(G)$ is an association scheme if and only if $G$ is *generously transitive*, that is, any two points of $\Omega$ are interchanged by some element of $G$.

When is there a non-trivial $G$-invariant association scheme?



## More definitions

The transitive permutation group $G$ is *AS-free* if the only $G$-invariant association scheme is the trivial scheme $\{I, J-I\}$.

The transitive permutation group $G$ is *AS-friendly* if there is a unique minimal $G$-invariant association scheme.

The transitive permutation group $G$ is *stratifiable* if we obtain an association scheme by *symetrising* $\mathcal{K}(G)$, that is, adding $A_i$ to $A_i^\top$ for non-symmetric $A_i$.

## Implications

The following implications hold between these conditions and others from permutation group theory.

| | | |
|---|---|---|
| $2$-transitive | $\Rightarrow$ | gen. trans. |
| $\Downarrow$ | | $\Downarrow$ |
| $2$-homogeneous | $\Rightarrow$ | stratifiable |
| $\Downarrow$ | | $\Downarrow$ |
| AS-free | $\Rightarrow$ | AS-friendly |
| $\Downarrow$ | | $\Downarrow$ |
| primitive | $\Rightarrow$ | transitive |

No implications reverse, and no more implications hold except possibly from “primitive” to “AS-friendly”.

All conditions in the table are closed under taking supergroups.



## An example

We describe a partition of $G^2$ invariant under right multiplication by giving a partition of $G$: to $C \subset G$ corresponds $\{(g,h) : gh^{-1} \in C\}$. We have a coherent configuration if and only if $\{1\}$ is a class and the class sums span a subring of the group ring (a *Schur ring*).

We get an association scheme if and only if each class is inverse-closed.

Let $G$ be the dihedral group

$$\langle a,b : a^3 = b^2 = (ab)^2 = 1\rangle = \{1,a,a^2,b,b,b^2\}.$$

Then $\{\{1\},\{a,a^2\},\{b\},\{ab,a^2b\}\}$ gives an association scheme, the $3 \times 2$ rectangle.

Similarly for $\{\{1\},\{a,a^2\},\{ab\},\{a^2b,b\}\}$ and $\{\{1\},\{a,a^2\},\{a^2b\},\{b,ab\}\}$.

But $\{\{1\},\{a,a^2\},\{b\},\{ab\},\{a^2b\}\}$ does not give an association scheme. So this group is not AS-friendly.



## Regular groups

For a regular permutation group $G$, the following are equivalent:

(a) $G$ is AS-friendly;

(b) $G$ is stratifiable;

(c) either $G$ is abelian, or $G \cong Q_8 \times A$, where $A$ is an elementary abelian $2$-group.

**Sketch proof:** (c) implies (b) by character theory; (b) implies (a) trivial; (a) implies (c) by an *ad hoc* argument using Dedekind’s Theorem.



## AS-free groups

An AS-free group is primitive, and is $2$-homogeneous, almost simple, or of diagonal type.

For, if imprimitive, it preserves the “group-divisible” association scheme; and, of the types in the O’Nan–Scott Theorem, groups of affine type are stratifiable, and groups of product type preserve Hamming schemes.

Obviously any $2$-homogeneous group is AS-free. Examples are known of almost simple AS-free groups which are not $2$-transitive, but they are quite hard to find. The smallest known example has degree $234$.

No examples of AS-free groups of diagonal type are known. Any such group must have at least four simple factors in its socle.

## Diagonal groups

Let $T$ be a group and $n$ a positive integer. Then $D(T,n)$ is the permutation group on the set $\Omega = \{[t_1,\ldots,t_n] : t_1,\ldots,t_n \in T\}$ generated by permutations of the following types:

- right translations by $T^n$;
- automorphisms of $T$ (acting in the same way on each coordinate);
- permutations of the coordinates;
- the map $\tau : [t_1,\ldots,t_n] \mapsto [t_1^{-1}, t_1^{-1}t_2, \ldots, t_1^{-1}t_n]$.

A diagonal group $D(T,n)$ is primitive if and only if $T$ is characteristically simple. If $T$ is simple, these are of diagonal type in the O'Nan–Scott Theorem; otherwise they are of product type.



## Diagonal groups with few simple factors

$D(T,1)$: we nave $\Omega = T$, and the diagonal group is generated by right translations, automorphisms, and inversion. If we just use inner automorphisms and no inversion, we obtain a coherent configuration (the corresponding Schur ring is spanned by the conjugacy class sums); this is commutative, so fusing inverse pairs gives an association scheme.

$D(T,2)$: We have $\Omega = T^2$. The matrix with $(t,u)$ entry $t^{-1}u$ is a Latin square. Any diagonal group preserves the corresponding *Latin square graph*.

So an AS-free diagonal group has at least four simple factors in its socle.



## General diagonal groups

We proved the following theorem:

(a) If $T$ is abelian then $D(T,n)$ is generously transitive.

(b) If $D(T,n)$ is generously transitive with $n \geq 8$, then $T$ is abelian.

(c) If $D(T,7)$ is generously transitive, then either $T$ is abelian, or $T \cong Q_8$.

(d) If $D(T,n)$ is stratifiable with $n \geq 9$, then $T$ is abelian.

Perhaps $9$ can be reduced to $8$ in part (d). This is best possible since $D(Q_8,7)$ is generously transitive.

We would like to have a similar bound for $n$ if $D(T,n)$ is AS-friendly!