

Association schemes and permutation groups

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Durham, July 2001

Joint work with P. P. Alejandro and R. A. Bailey

1

Permutation groups

Let G be a permutation group on Ω . Then the characteristic functions of the orbits of G on $\Omega \times \Omega$ form a coherent configuration $\mathcal{X}(G)$.

$\mathcal{X}(G)$ is an association scheme if and only if G is *generously transitive*, that is, any two points of Ω are interchanged by some element of G .

When is there a non-trivial G -invariant association scheme?

3

Definitions

A_0, \dots, A_{r-1} are $\Omega \times \Omega$ zero-one matrices. Such matrices represent subsets of $\Omega \times \Omega$.

Coherent configuration:

(a) $\sum_{i=0}^{r-1} A_i = J$, the all-1 matrix. (The corresponding subsets form a partition of $\Omega \times \Omega$.)

(b) $\sum_{i=0}^{s-1} A_i = I$. (The diagonal is a union of classes.)

(c) $A_i^\top = A_{i^*}$, where $*$ is an involution on $\{0, \dots, r-1\}$.

(d) $A_i A_j = \sum_{k=0}^{r-1} p_{ij}^k A_k$. (The matrices span an algebra.)

Association scheme: $*$ is the identity, i.e. all the relations are symmetric. (This implies that $s = 1$, that is, $A_0 = I$: the diagonal is a single class.)

2

More definitions

The transitive permutation group G is *AS-free* if the only G -invariant association scheme is the trivial scheme $\{I, J - I\}$.

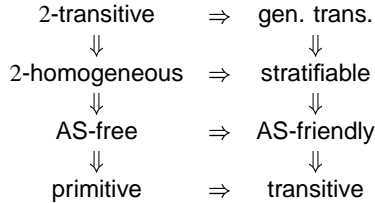
The transitive permutation group G is *AS-friendly* if there is a unique minimal G -invariant association scheme.

The transitive permutation group G is *stratifiable* if we obtain an association scheme by *symmetrising* $\mathcal{X}(G)$, that is, adding A_i to A_i^\top for non-symmetric A_i .

4

Implications

The following implications hold between these conditions and others from permutation group theory.



No implications reverse, and no more implications hold except possibly from “primitive” to “AS-friendly”.

All conditions in the table are closed under taking supergroups.

5

Regular groups

For a regular permutation group G , the following are equivalent:

- (a) G is AS-friendly;
- (b) G is stratifiable;
- (c) either G is abelian, or $G \cong Q_8 \times A$, where A is an elementary abelian 2-group.

Sketch proof: (c) implies (b) by character theory; (b) implies (a) trivial; (a) implies (c) by an *ad hoc* argument using Dedekind’s Theorem.

7

An example

We describe a partition of G^2 invariant under right multiplication by giving a partition of G : to $C \subset G$ corresponds $\{(g, h) : gh^{-1} \in C\}$. We have a coherent configuration if and only if $\{1\}$ is a class and the class sums span a subring of the group ring (a *Schur ring*).

We get an association scheme if and only if each class is inverse-closed.

Let G be the dihedral group

$$\langle a, b : a^3 = b^2 = (ab)^2 = 1 \rangle = \{1, a, a^2, b, b^2\}.$$

Then $\{\{1\}, \{a, a^2\}, \{b\}, \{ab, a^2b\}\}$ gives an association scheme, the 3×2 rectangle.

Similarly for $\{\{1\}, \{a, a^2\}, \{ab\}, \{a^2b, b\}\}$ and $\{\{1\}, \{a, a^2\}, \{a^2b\}, \{b, ab\}\}$.

But $\{\{1\}, \{a, a^2\}, \{b\}, \{ab\}, \{a^2b\}\}$ does not give an association scheme. So this group is not AS-friendly.

6

AS-free groups

An AS-free group is primitive, and is 2-homogeneous, almost simple, or of diagonal type.

For, if imprimitive, it preserves the “group-divisible” association scheme; and, of the types in the O’Nan–Scott Theorem, groups of affine type are stratifiable, and groups of product type preserve Hamming schemes.

Obviously any 2-homogeneous group is AS-free. Examples are known of almost simple AS-free groups which are not 2-transitive, but they are quite hard to find. The smallest known example has degree 234.

No examples of AS-free groups of diagonal type are known. Any such group must have at least four simple factors in its socle.

8

Diagonal groups

Let T be a group and n a positive integer. Then $D(T, n)$ is the permutation group on the set $\Omega = \{[t_1, \dots, t_n] : t_1, \dots, t_n \in T\}$ generated by permutations of the following types:

- right translations by T^n ;
- automorphisms of T (acting in the same way on each coordinate);
- permutations of the coordinates;
- the map $\tau : [t_1, \dots, t_n] \mapsto [t_1^{-1}, t_1^{-1}t_2, \dots, t_1^{-1}t_n]$.

A diagonal group $D(T, n)$ is primitive if and only if T is characteristically simple. If T is simple, these are of diagonal type in the O’Nan–Scott Theorem; otherwise they are of product type.

9

General diagonal groups

We proved the following theorem:

- If T is abelian then $D(T, n)$ is generously transitive.
- If $D(T, n)$ is generously transitive with $n \geq 8$, then T is abelian.
- If $D(T, 7)$ is generously transitive, then either T is abelian, or $T \cong Q_8$.
- If $D(T, n)$ is stratifiable with $n \geq 9$, then T is abelian.

Perhaps 9 can be reduced to 8 in part (d). This is best possible since $D(Q_8, 7)$ is generously transitive.

We would like to have a similar bound for n if $D(T, n)$ is AS-friendly!

11

Diagonal groups with few simple factors

$D(T, 1)$: we have $\Omega = T$, and the diagonal group is generated by right translations, automorphisms, and inversion. If we just use inner automorphisms and no inversion, we obtain a coherent configuration (the corresponding Schur ring is spanned by the conjugacy class sums); this is commutative, so fusing inverse pairs gives an association scheme.

$D(T, 2)$: We have $\Omega = T^2$. The matrix with (t, u) entry $t^{-1}u$ is a Latin square. Any diagonal group preserves the corresponding *Latin square graph*.

So an AS-free diagonal group has at least four simple factors in its socle.

10