An algebra related to enumeration

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Enumeration and orbit counting

Many combinatorial problems can be formulated as orbit-counting problems.

The relational structure M is *homogeneous* if every isomorphism between finite substructures of M can be extended to an automorphism of M.

The *age* of M is the class of all finite structures embeddable in M.

So, if M is homogeneous, the number of n-element structures in $\mathrm{Age}(M)$ (up to isomorphism) is equal to the number of orbits of $\mathrm{Aut}(M)$ on the set of n-subsets of M.

Fraïssé's Theorem

A class $\mathcal{C}$ of finite structures is the age of a countable homogeneous structure M if and only if

- $\mathcal{C}$ is closed under isomorphism;
- $\mathcal{M}$ is closed under taking induced substructures;
- $\mathcal{M}$ contains only countably many members up to isomorphism;
- $\mathcal{M}$ has the *amalgamation property*.

If these conditions hold, then M is unique up to isomorphism, and is called the *Fraïssé limit* of the *Fraïssé class* $\mathcal{C}$.

Thus, the enumeration problem for Fraïssé classes is equivalent to the orbit-counting problem for permutation groups.

Oligomorphic groups

The permutation group G on X is called *oligomorphic* if the number of G orbits on the set of n-subsets of X is finite for all n. (Equivalently, on X^n.)

The *Engeler–Ryll-Nardzewski–Svenonius Theorem* states that a countable first-order structure M is $\aleph_0$-categorical if and only if $\mathrm{Aut}(M)$ is oligomorphic.

Example: The ordered set $\mathbb{Q}$. (Cantor's Theorem characterises it as the unique countable dense ordered set without endpoints. Its automorphism group is transitive on n-sets for all n.)

Examples

Example 1. The finite graphs form a Fraïssé class. Its Fraïssé limit is the *random graph* or *Rado's graph* R.

Example 2. Consider the set of finite sets which are totally ordered and whose elements are coloured using the finite set A of colours. We can represent an n-element structure in this class as a word of length n in the alphabet A. The Fraïssé limit is the set $\mathbb{Q}$ with its elements coloured using the set A in such a way that each colour class is dense.

An algebra

Let X be an infinite set. For any non-negative integer n, let V_n be the set of all functions from the set of n-subsets of X to $\mathbb{C}$. This is a vector space over $\mathbb{C}$.

Define

$$A = \bigoplus_{n \geq 0} V_n,$$

with multiplication defined as follows: for $f \in V_m$, $g \in V_n$, let fg be the function in V_{m+n} whose value on the $(m+n)$-set A is given by

$$fg(A) = \sum_{\substack{B \subseteq A \\ |B|=m}} f(B)g(A \setminus B).$$

This is the *reduced incidence algebra* of the poset of finite subsets of X.

An algebra, continued

If M is a relational structure on X, let $A(M)$ be the subalgebra of A of the form $\bigoplus_{n\geq 0} V_n(M)$, where $V_n(M)$ is the set of isomorphism-invariant functions on the n-subsets of X.

If G is a permutation group on X, let A^G be the subalgebra of A of the form $\bigoplus_{n\geq 0} V_n^G$, where V_n^G is the set of functions fixed by G.

If M is homogeneous and $G = \mathrm{Aut}(M)$, then $A^G = A(M)$. In future we consider the group case, but most of this generalises.

If G is oligomorphic, then $\dim(V_n(G))$ is equal to the number $F_n(G)$ of orbits of G on n-sets.

The element e

Let e be the constant function in V_1 with value 1. Then it can be shown that e is not a zero-divisor. So multiplication by e is a monomorphism from V_n to V_{n+1}. In particular, $\dim(V_{n+1}^G) \geq \dim(V_n^G)$, that is, the sequence $(F_n(G))$ is non-decreasing.

Many results are known about the growth of this sequence. For example, Macpherson showed that, if G acts primitively, then either $F_n(G) = 1$ for all n, or the sequence grows at least exponentially.

A conjecture

Conjecture. If G has no finite orbits on X, then

(a) A^G is an integral domain;

(b) e is prime in A^G (that is, A^G/eA^G is an integral domain).

Conjecture (b) implies (a). These conjectures imply certain smoothness results about the growth of $(F_n(G))$, for example, (a) implies that

$$F_{m+n}(G) \geq F_m(G) + F_n(G) - 1.$$

Some results

We say that G is *entire* if A^G is an integral domain, and *strongly entire* if e is prime in A^G.

I can show the following.

- If G contains a (strongly) entire subgroup then G is (strongly) entire.
- If the point stabiliser, acting on the remaining points, is (strongly) entire, then G is (strongly) entire.

Polynomial algebras

Let M be the Fraïssé limit of $\mathcal{C}$. Under the following hypotheses, it can be shown that $A(M)$ is a polynomial algebra:

- there is a notion of *disjoint union* in $\mathcal{C}$;
- there is a notion of *involvement* on the n-element structures in $\mathcal{C}$, so that if a structure is partitioned, it involves the disjoint union of the induced substructures on its parts;
- there is a notion of *connected structure* in $\mathcal{C}$, so that every structure is uniquely expressible as the disjoint union of connected structures.

The polynomial generators of $A(M)$ are the characteristic functions of the connected structures.

Examples revisited

Example 1. If $\mathcal{C}$ is the class of finite graphs, let disjoint union and connectedness have their usual meaning, and let involvement mean spanning subgraph. The conditions are satisfied. So $A(R)$ is a polynomial algebra, where R is the random graph.

Example 2. let $\mathcal{C}$ consist of all finite words in the alphabet A. Let disjoint union mean concatenation in decreasing lexicographic order, involvement be lexicographic order reversed, and connected structures be the *Lyndon words* (those which are lexicographically smaller than their cyclic shifts). The conditions are satisfied. So the *shuffle algebra* on A is a polynomial algebra generated by the Lyndon words.

Transitive extensions

In each of the examples, the automorphism group of the infinite homogeneous structure has a transitive extension, which is strongly entire (by the earlier results). It is not known whether these algebras are polynomial algebras. These are interesting combinatorial problems.

Example 1. The transitive extension of $\mathrm{Aut}(R)$ is the automorphism group G of the countable homogeneous random *two-graph*. If A^G is a polynomial algebra, then the number of generators of dimension n is equal to the number of Eulerian graphs on n vertices.

Example 2. There are certain *circular structures* whose automorphism groups are transitive extensions of the groups of Example 2. They arise in model theory.