Some counting problems related to permutation groups

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'I count a lot of things that there's no need to count,' Cameron said. 'Just because that's the way I am. But I count all the things that need to be counted.'

Richard Brautigan, *The Hawkline Monster*

Three counting problems: 1

A *relational structure* M consists of a set X and a family of relations on X.

The *age* of M is the class of finite relational structures (in the same language) embeddable in M.

Problem. How many (a) *labelled*, (b) *unlabelled* structures in $\mathrm{Age}(M)$?

[Labelled structures have the element set $\{1,2,\ldots,n\}$. Unlabelled structures are isomorphism types.]

Three counting problems: 2

A permutation group G on a set X is *oligomorphic* if G has only finitely many orbits on X^n, for all n: equivalently, on the set of n-subsets of X, or on the set of n-tuples of distinct elements of X.

Problem. How many orbits on (a) n-sets, (b) n-tuples of distinct elements, (c) all n-tuples?

Three counting problems: 3

Let T be a complete consistent theory in the first-order language L. An *n-type* over T is a set S of formulae in L with free variables $x_1, \ldots, x_n$, maximal subject to being consistent with T.

We say that T is *$\aleph_0$-categorical* if it has a unique countable model (up to isomorphism). This is equivalent to there being only finitely many n-types for each n (the theorem of Engeler, Ryll-Nardzewski and Svenonius).

Problem. How many n-types?

An example

Let M be the unique countable dense totally ordered set $\mathbb{Q}$.

By *Cantor's Theorem*, its theory is $\aleph_0$-categorical.

Its age consists of all finite ordered sets: there is one unlabelled structure, and $n!$ labelled structures, on n elements.

Its automorphism group is transitive on n-sets for every n.

Connections: 12

The structure M is *homogeneous* if any isomorphism between finite induced substructures of M.

Fraïssé's Theorem: A class $\mathcal{C}$ of finite structures is the age of a countable homogeneous structure M if and only if it is closed under isomorphism, closed under taking induced substructures, contains only countably many members up to isomorphism, and has the *amalgamation property*.

If these conditions hold, then M is unique up to isomorphism. We call $\mathcal{C}$ a *Fraïssé class* and M its *Fraïssé limit*.

Connections: 12

There is a natural topology on the symmetric group of countable degree (pointwise convergence) with the properties that

(a) a subgroup is closed if and only if it is the automorphism group of a homogeneous relational structure;

(b) the closure of a subgroup is the largest overgroup with the same orbits on X^n for all n.

Hence counting labelled/unlabelled structures in a Fraïssé class is equivalent to counting orbits of a permutation group on n-sets/n-tuples of distinct elements.

Connections: 23

The theorem of Engeler, Ryll-Nardzewski and Svenonius says more than we have seen so far:

(a) for a countable structure M, the theory of M is $\aleph_0$-categorical if and only if $\mathrm{Aut}(M)$ is oligomorphic;

(b) if these condition holds, then all n-types are realised in M, and two n-tuples realise the same type if and only if they are in the same orbit of $\mathrm{Aut}(M)$.

Thus, if T is $\aleph_0$-categorical, counting n-types of T is equivalent to counting orbits of $\mathrm{Aut}(T)$ on n-tuples.

Three counting sequences

Let G be an oligomorhic permutation group on X. Let

$f_n(G) =$ no. of G-orbits on n-subsets;

$F_n(G) =$ no. of G-orbits on n-tuples of distinct elements;

$F_n^*(G) =$ no. of G-orbits on n-tuples.

Then f_n and F_n count unlabelled and labelled n-element structures in a Fraïssé class, while F_n^* counts n-types in an $\aleph_0$-categorical theory. We have

$F_n^* = \sum_{k=1}^{n} S(n,k)F_k$, where $S(n,k)$ is the Stirling number of the second kind;

$f_n \leq F_n \leq n!\, f_n$.

Three counting sequences

Which sequences occur? Let $\mathfrak{f}$ and $\mathfrak{F}$ be the sets of f- and F-sequences for oligomorphic groups. A compactness argument shows that both are closed in $\mathbb{N}^{\mathbb{N}}$ in the topology of pointwise convergence, so the conditions should be local ones!

Theorem: $f_{n+1} \geq f_n$ for all n. (Similarly $F_{n+1} \geq F_n$ but this is trivial.)

Example: total orders. $f_n = 1$, $F_n = n!$, and

$$F_n^* = \sum_{k=1}^{n} S(n,k)k!$$

is the number of *labelled preorders* on n points.

Growth rates: examples

Polynomial: for example, $f_n(S^k) = \binom{n+k-1}{k-1}$ is a polynomial of degree $k-1$ in n.

Fractional exponential: e.g. $f_n(S \,\mathrm{Wr}\, S) = p(n)$, the *partition function* (roughly $\exp(n^{1/2})$).

Exponential: e.g. for boron trees, $f_n \sim an^{-5/2}c^n$, where $c = 2.483\cdots$.

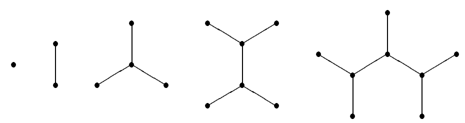
Another example: $f_n(S_2 \,\mathrm{Wr}\, A) = F_n$, the nth Fibonacci number.

Factorial: e.g. two independent total orders, $f_n = n!$.

Exponential of polynomial: e.g. graphs, $f_n \sim 2^{n(n-1)/2}/n!$.

Boron trees

A boron tree is a tree in which all vertices have valency 1 or 3. The leaves ('hydrogen atoms') of a boron tree carry a quaternary relation. The class of such relational structures is a Fraïssé class.



Growth rates: restrictions

Pouzet: For homogeneous binary relational structures, either

$c_1 n^d \leq f_n \leq c_2 n^d$ (for some $d \in \mathbb{N}$, $c_1, c_2 > 0$), or

f_n grows faster than polynomially.

Macpherson: In the latter case, $f_n > \exp(n^{1/2-\varepsilon})$ for $n > n_0(\varepsilon)$.

Macpherson: If G is primitive, then either $f_n = 1$ for all n, or $f_n > c^n$ for all sufficiently large n, where $c > 1$.

Smoothness

Sequences arising from groups should grow smoothly. In particular, for polynomial growth, $\log f_n / \log n$ should tend to a limit; for fractional exponential, $\log\log f_n / \log n$ for fractional exponential, $\log f_n / n$ for exponential, etc. *How do you state a general conjecture?*

A specific question. Define an operator S on sequences by $Sa = b$ if

$$\sum_{n=0}^{\infty} b_n x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-a_k}.$$

Is it true that, if $f = Sa$ counts orbits, then a_n / f_n tends to a limit (possibly 0 or 1)?

Smoothness

Remark 1. If $f = (f_n(G))$ then $Sf = (f_n(G \operatorname{Wr} S))$. Similar sequence operators can be defined with any oligomorphic group replacing S. The same conjecture could be made for any such operator. Similarly one could replace wreath products by direct products.

Remark 2. The operator S has various interpretations (see later).

An algebra

Let X be an infinite set. For any non-negative integer n, let V_n be the set of all functions from the set of n-subsets of X to $\mathbb{C}$. This is a vector space over $\mathbb{C}$.

Define

$$\mathcal{A} = \bigoplus_{n\geq 0} V_n,$$

with multiplication defined as follows: for $f \in V_m$, $g \in V_n$, let fg be the function in V_{m+n} whose value on the $(m+n)$-set A is given by

$$fg(A) = \sum_{\substack{B\subseteq A\\ |B|=m}} f(B)g(A\setminus B).$$

This is the *reduced incidence algebra* of the poset of finite subsets of X.

An algebra

If G is a permutation group on X, let $\mathcal{A}^G$ be the subalgebra of A of the form $\bigoplus_{n\geq 0} V_n^G$, where V_n^G is the set of functions fixed by G.

If G is oligomorphic, then $\dim(V_n^G)$ is equal to the number $F_n(G)$ of orbits of G on n-sets.

Integral domain?

I *conjecture* that, if G has no finite orbits, then $\mathcal{A}^G$ is an integral domain.

This would have as a consequence a smoothness result for the sequence (f_n), in view of the following result, in view of the following:

Let $\mathcal{A} = \bigoplus V_n$ be a graded algebra which is an integral domain, with $\dim(V_n) = a_n$. Then $a_{m+n} \geq a_m + a_n - 1$ for all m,n.

Polynomial algebra?

Let M be the Fraïssé limit of $\mathcal{C}$, and $G = \mathrm{Aut}(M)$.

Under the following hypotheses, it can be shown that $\mathcal{A}^G$ is a polynomial algebra:

- there is a notion of *disjoint union* in $\mathcal{C}$;

- there is a notion of *involvement* on the n-element structures in $\mathcal{C}$, so that if a structure is partitioned, it involves the disjoint union of the induced substructures on its parts;

- there is a notion of *connected structure* in $\mathcal{C}$, so that every structure is uniquely expressible as the disjoint union of connected structures.

The polynomial generators of $A(M)$ are the characteristic functions of the connected structures.

Polynomial algebra?

Note that:

- If the sequence $a = (a_n)$ counts the polynomial generators of degree n in a polynomial graded algebra, then Sa gives the dimensions of the homogeneous components;

- If the sequence $a = (a_n)$ counts connected structures in a class with a good notion of connectedness, then Sa counts arbitrary structures in the class.

A little problem

There is a unique countable homogeneous graph R containing all finite graphs. This is the *random graph* of Erdős and Rényi. Let $G = \mathrm{Aut}(R)$.

Since for graphs we have appropriate notions of connectedness and involvement, the algebra $\mathcal{A}^G$ is a polynomial algebra, whose generators correspond to connected graphs.

The group G has a transitive extension H, the automorphism group of the countable homogeneous universal two-graph.

[A *two-graph* is a collection $\mathcal{T}$ of 3-subsets of a set X having the property that any 4-subset of H contains an even number of members of $\mathcal{T}$.]

A little problem

Now it follows from general results that $\mathcal{A}^H$ is an integral domain.

Is it a polynomial algebra?

Mallows and Sloane showed that two-graphs and even graphs on n points are equinumerous (but there is no natural bijection).

Hence, if $\mathcal{A}^H$ is a polynomial algebra, then the number of polynomial generators of degree n is equal to the number of Eulerian (connected even) graphs on n vertices.