



LONDON  
MATHEMATICAL  
SOCIETY

# Mathematics: The next generation

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## A mathematics lecture

This is a maths lecture, so I will expect some engagement from you.

... mathematics is not best learned passively; you don't sop it up like a romance novel. You've got to go out to it, aggressive, and alert, like a chess master pursuing checkmate.

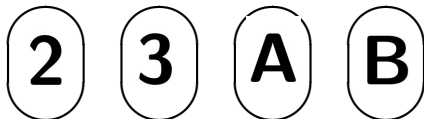
Robert Kanigel, *The Man who Knew Infinity: A Life of The Genius Ramanujan*

The reader should expect to make use of pen and paper in many places; mathematics is not a spectator sport!

Julian Havel, *Gamma: Exploring Euler's Constant*

## A problem

Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:



You have to test the following hypothesis:

*A card which has an even number on one side has a vowel on the other.*

You are allowed to turn over two cards. Which cards should you turn?

- ▶ 2 and A;
- ▶ 2 and B;
- ▶ a different pair;
- ▶ it's not possible.

# Mathematics is important!

Elliptic functions were introduced to measure the arc length of an ellipse. They led to

- ▶ Wiles' proof of Fermat's last theorem;
- ▶ elliptic curve cryptography, which keeps your transaction with an ATM secure.

Complex numbers were introduced to understand the process of solving cubic equations.

- ▶ They are essential in quantum mechanics, witness Born's equation

$$pq - qp = i\hbar.$$

- ▶ Quantum mechanics underlies the operation of all our electronic gadgets.

We cannot tell which bit of mathematics will be important next; so it is vital that we produce able and enthusiastic mathematicians!

# A good career

Them as counts counts moren them as dont count

Russell Hoban, *Riddley Walker*

From the Jobs Rated website (2009 data), out of 200 professions surveyed:

## 1. Mathematician

Applies mathematical theories and formulas to teach or solve problems in a business, educational, or industrial climate.

Overall Ranking: 1

Overall Score: 104

Work Environment: 89.720

Physical Demands: 3.97

Stress: 24.670

Income: \$94,160

Hours Per Week: 45

## Why major in mathematics?

One could say that mathematics is simply an **exciting**, **fascinating**, **utterly satisfying**, and **rapidly expanding** discipline. Many people are drawn to the level of **challenge** it presents, the **creativity** it requires, and the **clarity** it affords in knowing when you are right.

If you like **solving puzzles** and hunting for **patterns and hidden structures** – if you enjoy **logical analysis, deduction,** and **investigating the unknown** – if you want to understand the **connections** between seemingly widely different areas of science and technology, and how mathematics can be used to **explain** and **control** natural phenomena – then being a math major might be a good choice for you.

[Emphases mine]

But how do we get from here . . .

### OCR Specimen Exam Paper

Sketch the graph of  $y = \cos x^\circ$ , for values of  $x$  from 0 to 360.  
Sketch, on the same diagram, the graph of  $y = \cos(x - 60)^\circ$ .  
Use your diagram to solve the equation

$$\cos x^\circ = \cos(x - 60^\circ)$$

for values of  $x$  between 0 and 360. Indicate clearly on your diagram how the solutions relate to the graphs.  
State how many values of  $x$  satisfying the equation

$$\cos(10x)^\circ = \cos(10x - 60)^\circ$$

lie between 0 and 360. (You should explain your reasoning briefly, but no further detailed working or sketching is necessary.)



... to here?

Annals of Mathematics, 142 (1995), 443–551

# Modular elliptic curves and Fermat's Last Theorem

By ANDREW WILES\*

*For Nada, Clare, Kate and Olivia*

*Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere: cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*

*Pierre de Fermat*

## The problem

At school, because of league tables, pupils and teachers both have a vested interest in maximising exam marks (especially at the C/D borderline); the examiner is the guardian at the end of the level.

At university, lecturers want the pupils to understand the material, and to learn to work independently. They are, incidentally, also the examiners.

We are more interested in helping the students to *understand* what they are doing than to cram stuff into short-term memory for the exams.

On the other hand, there is so much mathematics to know; there are voices saying “You can’t call yourself a maths graduate unless you know about Lebesgue measure/finite simple groups/the Index Theorem/sheaves/...”.

## Our response



Part of our response to help students make this transition was to develop a new first-semester module, **MTH4110 Mathematical Structures**, for students taking maths or joint degrees including maths, at Queen Mary, University of London. The hope was that it would teach mathematical thinking and understanding, and give the students good habits, which would transfer to their other modules.

I was given the job of producing and delivering this module. I would like to thank Thomas Prellberg, whose vision led to this, and whose unwavering support was crucial to its success.

## The specification (extract)

- ▶ It should be a first-semester module, **compulsory** for mathematics students (including those on joint programmes), designed to introduce them to **rigorous mathematical thinking** and **fundamental objects** such as sets, functions, and numbers.
- ▶ Despite this, rigorous mathematical thinking, accuracy and understanding are important in all mathematics modules; all first-semester lecturers will be expected to convey this message to the students they teach.

I was nervous of putting “rigorous mathematical thinking” into a ghetto, hence the second clause.

## What was in the course?

The ten chapters were as follows:

- ▶ Introduction
- ▶ Sets
- ▶ Infinity
- ▶ Functions and relations
- ▶ Natural numbers
- ▶ Integers and rational numbers
- ▶ Real numbers
- ▶ Complex numbers
- ▶ Proofs
- ▶ Constructing and debugging proofs

The intention was to keep the content minimal, to allow time for discussion of how to do mathematics.

## Supplementary material

With each of the ten chapters in the notes, I also produced supplementary material, some of it historical and philosophical, some developing the material further, and some containing more examples to round out the notes.

Each chapter of the notes also included some advice on study skills, from the advice to refer to your notes for a definition if there is a word you don't understand in the question, to what to do in the exam room.

I wanted everything open and transparent, so course material was kept on a web page visible to the public, and I commented on the progress of the course on my blog "Cameron Counts".

I collected interesting examples and problems from many colleagues and friends. Special thanks to Chris Budd!

## Details

In my experience, one of the worst breakers of mathematical concentration is coming to a symbol which you don't know how to pronounce. What would you make of  $\zeta \otimes \eta$ ?

As a student, I came to grief repeatedly over the Fraktur capitals  $\mathfrak{G}$ ,  $\mathfrak{S}$ , and the rest of them, in a book I was reading. So I provided the students with a table of the Greek alphabet with the names of the letters written out, and for every new kind of formula, notes about how to read it.

There was also advice to students on how to write blackboard bold characters  $\mathbb{N}$ ,  $\mathbb{Z}$ , etc., by hand in their notes (and why we use the particular letters we do), the difference between  $a/b$  and  $a \mid b$ , and so on.

# Tutorials

One very important aspect was the re-introduction (after many years absence at Queen Mary) of small-group tutorials (five or six students and a tutor).

In recent years, students have chosen modules on-line and have tended to have little interaction with their academic advisers. We addressed this as well, by arranging that the students' tutor was their academic adviser.

The tutorials were very successful (as I will discuss later).



## What's it all about?

A new module has to have learning outcomes, key objectives, and all that (or whatever the jargon is now), and this one did. I was pleased to be able to get a quote from T. S. Eliot into the course description: “precise but not pedantic”.

But on the web page is a description of what the module was really about.

The module has three main aims:

1. to introduce the basic objects of mathematics (numbers, sets and functions), and their properties;
2. to emphasize the fact that mathematics is concerned with proofs, which establish results beyond doubt, and to show you how to construct proofs, how to spot false “proofs”, how to use definitions, etc.;
3. to get you involved in the excitement of doing mathematics.

Of course the third is the most important!

## Re-inventing the wheel?

There is material available which aims at some of the same targets. Mostly, I deliberately chose not to read it, since I wanted to find my own approach.

Honorable mention to Kevin Houston for his book *How to Think like a Mathematician*. It is supported by a short PDF document, "Ten ways to think like a mathematician."

I was delighted to find that the first two of the ten are:

1. Question everything. [Don't take anyone else's word for it, or as the Royal Society has it, *Nullius in Verba*. It's true for all scientists, but far more so for mathematicians.]
2. Write in sentences. [One of the commonest mistakes students make is to write a chain of formulae connected with = or  $\Leftrightarrow$  signs and expect to get full marks.]

## Entering a mathematics department



I was an undergraduate at the University of Queensland. The Mathematics Department was on the side of a hill. You enter on the ground floor, and can go down to the basement or up to the higher floors.

When you study mathematics at university, you enter with your existing knowledge (both explicit and tacit) of numbers (both integers and real numbers) and space. You can go down to logic and set theory, or up to analysis and group theory. My aim was to build on this, not to tear everything up and start again with abstractions. This decision had several consequences.

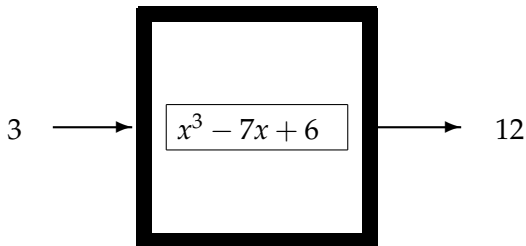
## Sets and functions

Philosophers have not found it easy to sort out sets ...

D. M. Armstrong, *A Combinatorial Theory of Possibility*

I did not attempt to define sets.

A function was, not a set of ordered pairs, but a black box where you put in an element of the domain and get out its image. The name of the function is written on the box, and the domain and codomain are part of the specification. If the black box jams when you put in an element of the domain, or if the output is not in the codomain, ask for your money back!



## Natural numbers

Again, I assume that everyone knows what natural numbers are. The key property is

*You can, in principle, start at 1 and count up to any natural number.*

When my children were in primary school, there was a vogue for a rhyme that went:

*One, two  
Missed a few,  
Ninety-nine,  
A hundred.*

Or, as a mathematician would say,  $1, 2, \dots, 99, 100$ .

## Integers and rational numbers

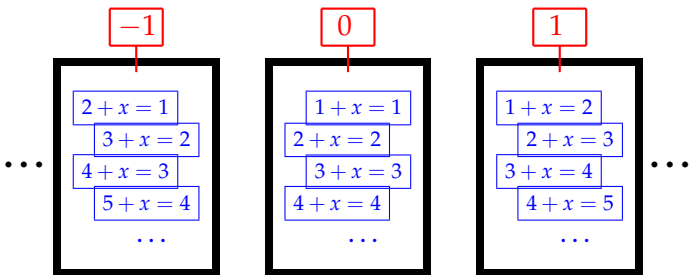
Again, I assumed these known.

However, I did give (in the notes, not the lectures) a definition of the integers in terms of the natural numbers.

The naive way to do this is to say: an integer is either 0 or a natural number with  $+$  or  $-$  in front of it. The trouble with this definition is that, to define addition of integers, you have to consider thirteen separate cases!

The mathematician's definition of an integer is an equivalence class of ordered pairs of natural numbers. The pair  $(a, b)$  represents the integer  $b - a$ . But we don't think of integers this way ...

I pictured an integer as a bag full of equations of the form  $b + x = a$ , all defining the same integer; to do calculations with integers, you don't have to look inside the bag, just read the label on the front!



Now the equation  $b + x = a$  is a way of giving meaning to the ordered pair  $(a, b)$ , and we can define addition and multiplication without cases.

## Real numbers

Mathematicians define real numbers by either Dedekind cuts or Cauchy sequences, based on the rational numbers. Neither of these is suitable for a beginner (who in any case also has a good intuition about real numbers).

So I defined real numbers as infinite decimals.

It's easy to prove that the set of real numbers is uncountable, that every positive real number has a real square root, and so on.

There are some difficulties, which I simply skipped over. For example, consider

$$x = 0.386732054789 \dots + 0.613467945210 \dots$$

Is  $x$  greater than, equal to or less than 1?



## Reasoning and logic

Reasoning and logic are to each other as health is to medicine, or — better — as conduct is to morality. Reasoning refers to a gamut of natural thought processes in the everyday world. Logic is how we ought to think if objective truth is our goal — and the everyday world is very little concerned with objective truth. Logic is the science of the justification of conclusions we have reached by natural reasoning. My point here is that, for such natural reasoning to occur, consciousness is not necessary. The very reason we need logic at all is because most reasoning is not conscious at all.

Julian Jaynes, *The Origin of Consciousness in the Breakdown of the Bicameral Mind*

## Implication

In non-mathematical discourse, “A implies B” suggests that there is a causal connection between A and B.

Mathematicians, instead, say that “A implies B” holds in any situation except where A is true and B is false.

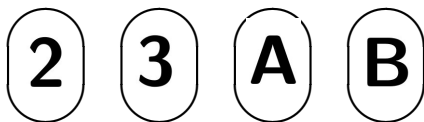
Suppose I say to you,

*If it's fine tomorrow, I'll take you to the Zoo.*

The only situation where I have lied is if it is fine and I don't take you to the Zoo. If it rains, my statement is not wrong, no matter what we do.

## A problem revisited

Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:



You have to test the following hypothesis:

*A card which has an even number on one side has a vowel on the other.*

The hypothesis is false only if there is a card with an even number on one side and a consonant on the other. So we have to check cards 2 and B.

## Proof by contradiction

The proof [of the existence of an infinity of prime numbers] is by *reductio ad absurdum*, and *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's favourite weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.

G. H. Hardy, *A Mathematician's Apology*

If the assumption that A is false leads us to an impossible or nonsensical situation, then we know that A must be true.

# Euclid's proof

## Theorem

*There are infinitely many prime numbers.*

## Proof.

Suppose that there are only finitely many prime numbers, say  $p_1, \dots, p_n$ . Let  $N$  be the number obtained by multiplying them all together and adding 1. Then  $N$  is bigger than all of  $p_1, \dots, p_n$ , so it can't itself be prime, and it must have a prime divisor, necessarily in this set (since these are all the primes). But every prime in this set leaves a remainder of 1 when divided into  $N$ . □

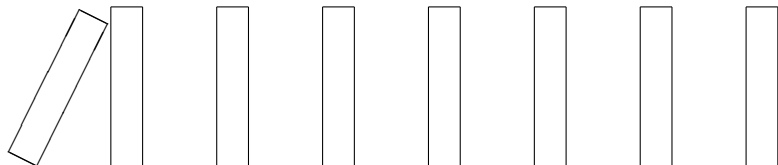
## A proof is a convincing argument

When I gave this proof, I told the students that a proof is meant to be a convincing argument; if they were not convinced, I had not done my job, and they should ask questions. Many of them took me up on this, and continued to do so throughout the course.

There is a weak spot in the proof. Why does  $N$  have to have a prime divisor at all? The fact that every natural number greater than 1 has a prime divisor needs to be proved, and the proof of this depends on the **most important** property of the natural numbers, **Induction**.

## Induction

Suppose we have a long (potentially infinite) line of dominoes:



Suppose we have a **guarantee** that, when any domino falls, it will knock over the next domino. What happens when I push over the first domino?

They all fall over!

This is the **Principle of Induction**.

## More formally

*Suppose that  $P(n)$  is some statement about the natural number  $n$ . Suppose that*

- ▶  *$P(1)$  is true;*
- ▶  *$P(n)$  implies  $P(n + 1)$ , for any natural number  $n$ .*

*Then  $P(n)$  is true for all natural numbers  $n$ .*

On the previous slide, let  $P(n)$  be the statement “domino number  $n$  falls over”. The second bullet point is the **guarantee** I gave you, and the first corresponds to my pushing over the first domino.

But compare the two statements, on the last slide and this one. The first is to be *understood*, the second to be learned and trotted out in an exam.



## Why is it true?

Remember the property of the natural numbers: I can start at 1 and count up to any number. Property  $P$  holds at the start, and remains true at every step of the count; so it is true at the end. One problem that beginning students have is that they are told “Assume  $P(n)$  and prove  $P(n + 1)$ .” But we are trying to prove  $P(n)$ ; are we not assuming what we are required to prove? My hope is that, once you understand induction, this is no longer a problem.

The statement that every natural number greater than 1 has a prime factor (which we needed for Euclid’s proof) can be proved by induction. But I won’t inflict the proof on you now. (It is an “exercise for the reader”.)

## False proofs

As Julian Jaynes pointed out, sometimes arguments appear plausible but contain logical flaws which can only be discovered by extreme care.

### Theorem

*All horses have the same colour.*



This is a proof by induction. Let  $P(n)$  be the statement:

*In any set of  $n$  horses, all the horses have the same colour.*

We prove  $P(n)$  by induction.

## Proof.

First, to start the induction,  $P(1)$  is obviously true; in a set containing only one horse, clearly all the horses in the set have the same colour!

Next, the inductive step: We assume that  $P(n)$  is true and prove  $P(n + 1)$ . Accordingly, let  $\{H_1, H_2, \dots, H_{n+1}\}$  be a set of  $n + 1$  horses. Then

- ▶ the subset  $\{H_1, H_2, \dots, H_n\}$  is a set of  $n$  horses, so horses  $H_1, H_2, \dots, H_n$  all have the same colour, by the induction hypothesis;
- ▶ the subset  $\{H_2, \dots, H_n, H_{n+1}\}$  is also a set of  $n$  horses, so that the horses  $H_2, \dots, H_n, H_{n+1}$  all have the same colour, again by the induction hypothesis.

It follows that all of  $H_1, \dots, H_{n+1}$  have the same colour, and so the inductive step is complete, and with it the proof. □

## Tutors' comments

There were some teething troubles, but the tutors very quickly became engaged. I tried to provide interesting questions which would provoke discussion.

At the start of the semester, two of my colleagues had a bet, one claiming that the students would stop turning up at tutorials by the mid-term.

I am happy to report that the cynic lost the bet! Tutors were reporting full attendance, or emailed apologies in advance, at the end of the semester.

## Tutors' comments

### Typical comment:

Good engagement with the group in the tutorial. But reluctance to get involved - "can't see how to write it down for case "n" when they argued it perfectly for "n=4" re. no two people have the same number of friends. Have asked them to hand in the solutions we discussed for me to look at - hope this doesn't break too many rules? I had 3 of the 6 stand up and write on the board which wasn't bad. Generally nice problems to talk around in Ex 1 and 2. I enjoyed the tutorial.

## Student comments: positive

- ▶ Different view of maths than I had previously encountered.
- ▶ Professor Cameron makes the lectures interesting by adding some history/philosophical thoughts in.
- ▶ Tutorials are very helpful, small classes enable a lot of learning to be done. Online notes are useful.
- ▶ Intervals at the appropriate times for questions.
- ▶ The lecturer ... makes the students interact more than any other lecturer.
- ▶ The professor ... is passionate about mathematics.

There were **many** positive comments about the small group tutorials.

## Student comments: negative

This section asked how the module could be improved. Every lecturer knows that there will be many students who ask for “more examples”. On the other hand, giving more examples without the underlying theory does not help understanding. Many students asked for more tutorials. Clearly they wanted the small group tutorials extended to other modules.

Unfortunately this is not practicable!

Students asked for notes to be put online in advance of the lectures. Rightly or wrongly, I do not do this. I believe that taking notes is an important part of learning and helps get the material into the students' brains.

## An email

Dear Professor Cameron,

I'd just like to say a HUGE thank you! You made the transition from A-level to University much easier for me, you taught me to look at mathematics in a different way and I feel I am now able to approach this course in the way I should after learning Mathematical Structures. Every single lecture you gave was so intriguing and I didn't realise how much I had learnt from them until the mid-term exam.

Approaching the Midterms, Mathematical Structures was the one I was most worried about as it required a different approach compared to Calculus, Probability and Mathematical Computing, but after reading through my notes I realised that I actually knew most of the material and ended up getting 97/100 (which was my highest mark!).



I would just like to say thank you one more time, you have made me very confident in knowing that I can achieve my best at Queen Mary and in Mathematics as a whole and I think I can speak for everyone on the course when I say I hope you were able to stay and teach all of our modules! Hope you have a lovely time in Portugal and a great Christmas!

Vickie Weller

So the course was a success!