Sudoku, Mathematics and Statistics

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Forder lectures
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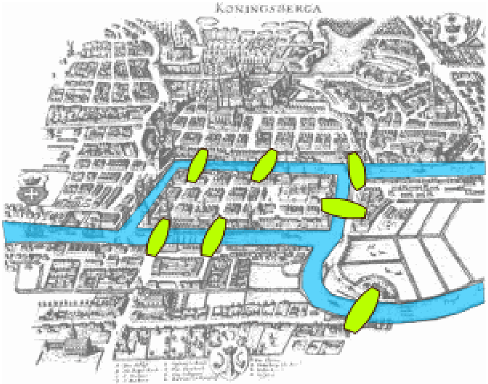
There's no mathematics involved. Use logic and reasoning to solve the puzzle.

Instructions in *The Independent*

Euler

The bridges of Königsberg



Is it possible to walk around the town, crossing each bridge exactly once?

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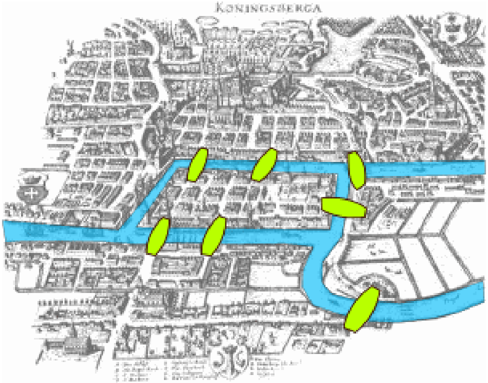

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Euler showed: No!

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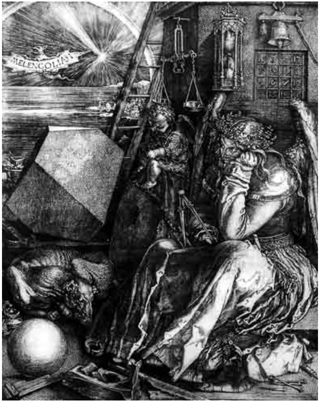
>
> *In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way.*

Dürer's *Melancholia*



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

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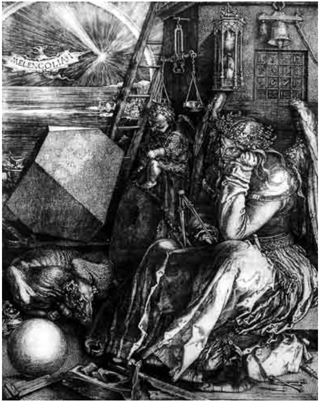

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All rows, columns, and diagonals sum to 34.

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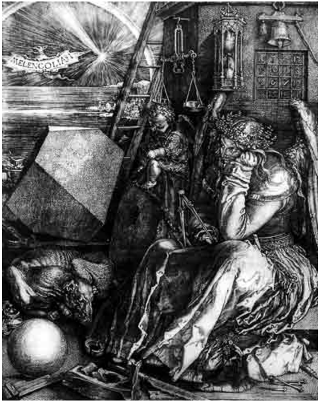

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All rows, columns, and diagonals sum to 34. The date of the picture is included in the square.

Euler's construction

Take a Graeco-Latin square of order n.

$C\beta$	$A\gamma$	$B\alpha$
$A\alpha$	$B\beta$	$C\gamma$
$B\gamma$	$C\alpha$	$A\beta$

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21	02	10
00	11	22
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So for which n do Graeco-Latin squares exist?

Euler's officers

Six different regiments have six officers, each one holding a different rank (of six different ranks altogether). Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one from each regiment?

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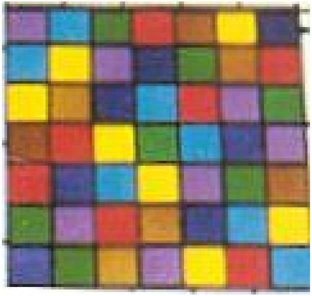
> *Six different regiments have six officers, each one holding a different rank (of six different ranks altogether). Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one from each regiment?*

Trial and error suggests the answer is "No":

Latin squares in statistics



Latin squares were introduced into statistics by R. A. Fisher.

Latin squares in statistics

A Latin square at Rothamsted Experimental Station.

This Latin square was designed by Rosemary Bailey. Thanks to Sue Welham for the photograph.

Gerechte designs

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1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

Gerechte designs

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1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

This is a gerechte design (a “fair design”).

Critical sets

John Nelder: A critical set is a partially filled Latin square which can be completed in a unique way to a Latin square, but if any entry is deleted the completion is no longer unique.

1	2		
2			
			3

Sudoku

So a Sudoku puzzle is a partial gerechte design for the partition of a 9×9 square into nine 3×3 subsquares, which contains a critical set.

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In fact Sudoku was invented by Howard Garns (a retired New York architect) in the 1980s, under the name "number place", was taken up in Japan and re-names Sudoku, and spread worldwide from there.

How many Sudoku solutions?

We count Sudoku solutions up to

- Permuting the numbers $1, \ldots, 9$;
- Permuting rows and columns preserving the partitions into 3 sets of 3;
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This was computed by Jarvis and Russell using the Orbit-counting Lemma applied to the group $(S_3 \operatorname{wr} S_3) \operatorname{wr} S_2$ of order $6^8 \cdot 2$.

Robert Connelly's Symmetric Sudoku

Each number from 1 to 9 should occur once in each set of the following types:

- rows;
- columns;
- 3×3 subsquares;
- broken rows (one of these consists of three "short rows" in the same position in the three subsquares in a large column);
- broken columns (similarly defined);
- locations (a location consists of the nine cells in a given position, e.g. middle of bottom row, in each of the nine subsquares).

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Rows

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Columns

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Subsquares

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Broken rows

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Broken columns

Example

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
$\boxed{7}$	2	6	$\boxed{9}$	1	5	$\boxed{8}$	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
$\boxed{5}$	9	3	$\boxed{4}$	8	2	$\boxed{6}$	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
$\boxed{1}$	4	8	$\boxed{3}$	6	7	$\boxed{2}$	5	9

Locations

Affine geometry

We coordinatise the cells of the grid with F^4, where F is the integers mod 3, as follows:

- the first coordinate labels large rows;
- the second coordinate labels small rows within large rows;
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Now Connelly's regions are cosets of the following subspaces:

Rows	$x_1 = x_2 = 0$	Columns	$x_3 = x_4 = 0$
Subsquares	$x_1 = x_3 = 0$	Broken rows	$x_2 = x_3 = 0$
Broken columns	$x_1 = x_4 = 0$	Locations	$x_2 = x_4 = 0$

Affine spaces and SET

The card game SET has 81 cards, each of which has four attributes taking three possible values (number of symbols, shape, colour, and shading). A winning combination is a set of three cards on which either the attributes are all the same, or they are all different.

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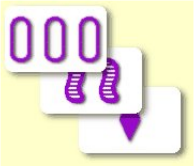

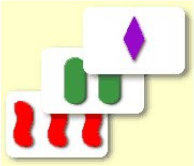

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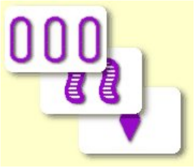 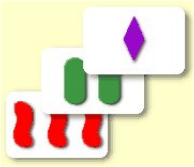

Each card has four coordinates taken from F (the integers mod 3), so the set of cards is identified with the 4-dimensional affine space. Then the winning combinations are precisely the affine lines!

Perfect codes

A code is a set C of "words" or n-tuples over a fixed alphabet F. The Hamming distance between two words v, w is the number of coordinates where they differ; that is, the number of errors needed to change the transmitted word v into the received word w.

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A code C is e-error-correcting if there is *at most* one word at distance e or less from any codeword. [Equivalently, any two codewords have distance at least $2e + 1$.] We say that C is perfect e-error-correcting if "at most" is replaced here by "exactly".

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- So the entire solution is a partition of the affine space into nine perfect codes.
- Using the SET test, a perfect code is an affine subspace.
- So there are only two different symmetric Sudoku solutions.

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- not too much slower (in terms of number of computation steps required) than any computer yet built.

The last statement would become false if a quantum computer were to be built.

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A type of problem is non-deterministic polynomial time, or in the class NP, if we can recognise a solution in a polynomial number of steps.

Complexity measures

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In other words, if a problem is NP-complete, then no known algorithm solves it efficiently, and we suspect that no efficient solution can exist.

Example

Given a graph G,

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- the problem of walking round the graph so that every edge is traversed exactly once is in P (this is Euler's bridges of Königsburg in disguise);
- the problem of walking round the graph so that each vertex is visited exactly once is NP-complete (this is Hamilton's Icosian game).

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- Is there a test for completability short of actually trying to do it?
- What about finding an orthogonal mate to a Latin square (one which extends it to a Graeco-Latin square)?