3 Fatou and Julia sets

The following properties follow immediately from our definitions at the end of the previous chapter:

1. $F(f)$ is open (by definition); hence $J(f)$ is closed and therefore compact (since $\hat{\mathbb{C}}$ is compact).

2. $F(f)$ is *completely invariant*, that is $f(F(f)) = F(f) = f^{-1}(F(f))$. The fact that $f^{-1}(F(f)) \subset F(f)$ follows from the definition of $F(f)$ and the continuity of f; the converse, $F(f) \subset f^{-1}(F(f))$, is a consequence of the fact that a rational map is *open* (i.e the image of an open set is itself open).

3. $J(f)$ is completely invariant. (This follows at once from 2.)

What kinds of families $\mathcal{F}$ of analytic maps $f : \Omega \to \hat{\mathbb{C}}$ are equicontinuous ? Our first step towards an answer is the following very useful Proposition which interprets Schwarz's Lemma in the language of hyperbolic geometry:

Proposition 3.1 *If f be a holomorphic map $\mathbb{D} \to \mathbb{D}$ then f is non-increasing in the Poincaré metric.*

Proof Let z_0, z_1 be any two points in $\mathbb{D}$. Let $f(z_0) = w_0$ and $f(z_1) = w_1$. Choose isometries h, k of the Poincaré disc D (Möbius transformations) such that $h(0) = z_0$ and $k(0) = w_0$. Let $z_1' = h^{-1}z_1$ and $w_1' = k^{-1}w_1$. Now $k^{-1}fh$ is a holomorphic map of $\mathbb{D}$ to itself sending 0 to 0 and z_1' to w_1'. Hence $|w_1'| \leq |z_1'|$ by Schwarz's Lemma, and so $d(0, w_1') \leq d(0, z_1)$ in the Poincaré metric. But $d(w_0, w_1) = d(0, w_1')$ and $d(z_0, z_1) = d(0, z_1')$ (as h and k are isometries). QED

Corollary 3.2 *Every family of holomorphic maps $\mathbb{D} \to \mathbb{D}$ is equicontinuous.*

Proof It follows at once from Proposition 3.1 that every such family $\mathcal{F}$ is equicontinuous with respect to the Poincaré metric on $\mathbb{D}$. But we need to show that it is equicontinuous with respect to the spherical metric (where we regard $\mathbb{D}$ as $\mathbb{D} \subset \mathbb{C} \subset \hat{\mathbb{C}}$). However given any point z_0 we can find a small disc around z_0 and a constant k such that the distance between any two points z, z' in this disc in the Poincaré metric is less than k times the distance in the spherical metric. Equicontinuity at z_0 follows, since the spherical distance between the images $f(z), f(z')$ of two points under $f \in \mathcal{F}$ is less than or equal to the Poincaré distance betwen these images, this being true for any pair of points in $\mathbb{D}$. QED

Example The family $\{z \to z^{2^n}\}_{n\geq 0}$ is equicontinuous on $\mathbb{D}$: thus the Fatou set of $z \to z^2$ contains $\{z : |z| < 1\}$. Conjugating by $\sigma : z \to 1/z$ we see that the Fatou set of $z \to z^2$ also contains $\{z : |z| > 1\}$. Since every point on the unit circle is in the Julia set of $z \to z^2$, we now have a proof that the Fatou and Julia sets of this map are as we claimed at the end of the previous chapter.

It follows at once from Corollary 3.2 that every *bounded* family of holomorphic maps $\mathbb{D} \to \mathbb{C}$ is equicontinuous (again with respect to the spherical metric).

There are two approaches to defining the Fatou set of a rational map f, either as the *equicontinuity set* of the family of iterates of f, or as the *normality set* of this family. They give equivalent definitions, so it really makes no difference which route we take, but it will be convenient for us to switch back and forth.

Definition Let Ω be a domain in $\hat{\mathbb{C}}$. A family $\mathcal{F}$ of maps $\Omega \to \hat{\mathbb{C}}$ is called *normal* if every sequence in $\mathcal{F}$ contains a subsequence which converges *locally uniformly* to a map $f : \Omega \to \hat{\mathbb{C}}$ (not necessarily in $\mathcal{F}$).

Example $\{z \to z^{2^n}\}_{n\geq 0}$ are a normal family on $\mathbb{D}$, since they converge locally uniformly there to the constant map $z \to 0$.

Theorem 3.3 (Arzelà-Ascoli) *Let Ω be a domain in $\hat{\mathbb{C}}$. Any family of continuous maps $\Omega \to \hat{\mathbb{C}}$ is normal if and only if it is equicontinuous.*

For a proof, see for example Ahlfors' book 'Complex Analysis'.

Comments

1. We remind the reader that we use the *spherical metric* on $\hat{\mathbb{C}}$, both in the definition of *local uniform convergence* (used in defining the notion of a *normal family*) and in the definition of *equicontinuity*.

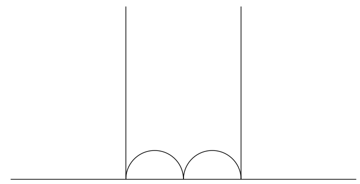

Figure 5: A fundamental domain Δ for the action of the group G on the upper half-plane (Theorem 3.4). The two vertical lines are identified by $z \to z+1$, the two semicircles are identified by $z \to z/(2z+1)$, and the quotient Δ/G is a thrice punctured sphere.

2. It is more elegant mathematically to develop the whole Fatou-Julia theory via normality rather than equicontinuity but the latter is perhaps easier to comprehend dynamically (Milnor follows the normality route).

3. It follows at once from Corollary 3.2 and Theorem 3.3 that every family of holomorphic maps from $\mathbb{D}$ to itself is normal. One can also prove Corollary 3.2 directly from the definition of a normal family (see Milnor, Thm 3.2, for a general version): at the heart of the argument is the Bolzano-Weierstrass Theorem that in any metric space a set is compact if and only if every infinite subset contains a convergent subsequence. Also relevant to this circle of ideas is the Denjoy-Wolff Theorem (1926), which states that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is either a conformal bijection or the iterates of f converge locally uniformly to a constant map $\mathbb{D} \to \zeta \in \bar{\mathbb{D}}$.

This brings us to a theorem central to the development of the Fatou-Julia theory of rational maps:

Theorem 3.4 (Montel, 1911) *let Ω be a domain in $\hat{\mathbb{C}}$. Every family of analytic maps $\Omega \to \hat{\mathbb{C}} - \{0, 1, \infty\}$ is normal (or equivalently, by Arzelà-Ascoli, equicontinuous).*

Proof Without loss of generality assume Ω is an open disc (since equicontinuity and normality are local properties), and indeed by scaling if necessary assume $\Omega = \mathbb{D}$, the unit disc. Since $\mathbb{D}$ is simply connected, any map $f : \mathbb{D} \to \hat{\mathbb{C}} - \{0, 1, \infty\}$ lifts to a map $\tilde{f}$ from $\mathbb{D}$ to the *universal cover* of $\hat{\mathbb{C}} - \{0, 1, \infty\}$, which is the complex upper half plane $\mathcal{H}_+$, the group of covering translations being

$$G = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

acting on the upper half plane in the usual way (Figure **??**).

*[**Aside** Here we recall that a universal cover of a manifold M is a simply-connected manifold $\tilde{M}$ which evenly covers M, i.e. $\tilde{M}$ is equipped with a projection $p : \tilde{M} \to M$ with the property that every $x \in M$ has a neighbourhood U such that $p^{-1}(U)$ is a disjoint union of copies of U, each mapped homeomorphically by p onto U. Given a universal cover $p : \tilde{M} \to M$ and a simply-connected space Y, every continuous $f : Y \to M$ has a lift, $\tilde{f} : Y \to \tilde{M}$ such that $p\tilde{f} = f$, and indeed there is a unique $\tilde{f}$ that lifts a chosen base point $x \in M$ to any specified point in $p^{-1}(x)$.]*

Equivalently we may take the universal cover to be the Poincaré disc $\mathbb{D}$. Observe that the set of all lifts $\tilde{f} : \mathbb{D} \to \mathbb{D}$ of elements f of $\mathcal{F}$ forms a normal family, since these lifts are self-maps of the disc. The Poincaré metric on $\mathbb{D}$ projects under $q : \mathbb{D} \to \hat{\mathbb{C}} - \{0, 1, \infty\}$ to a metric on $\hat{\mathbb{C}} - \{0, 1, \infty\}$ in which the three missing points are pairwise infinitely far apart. Taking the Poincaré metric on domain and range each $\tilde{f}$ is metric non-increasing (by Proposition 3.1) and hence the same is true for each $f : \mathbb{D} \to \hat{\mathbb{C}} - \{0, 1, \infty\}$. Given any $z_0 \in \mathbb{D}$, we may restrict consideration to a small disc $D' \subset \mathbb{D}$ centred on z_0 (since equicontinuity and normality are local properties). Since D' has finite diameter, say k (in the Poincaré metric), each $f(D')$ has diameter $\leq k$, and so for a small disc neighbourhood N of at least one of the three missing points in $\hat{\mathbb{C}} - \{0, 1, \infty\}$, there must be an infinite sub-family $\mathcal{F}' \subset \mathcal{F}$ such that $f(D') \cap N = \emptyset$ for all $f \in \mathcal{F}'$. Since $\hat{\mathbb{C}} - N$ is a disc, the family $\mathcal{F}'$ is equicontinuous with respect to the spherical metric (by Prop. 3.2), therefore normal, and hence so is $\mathcal{F}$. QED

We can replace the points $0, 1, \infty$ in the statement of Montel's Theorem by any other three points of $\hat{\mathbb{C}}$ (just compose with a suitable Möbius transformation). Montel's Theorem is a much more powerful result than our earlier observation that any family of maps with a common bound is equicontinuous. One should perhaps compare it with Picard's Theorem that any holomorphic function $\mathbb{C} \to \hat{\mathbb{C}} - \{0, 1, \infty\}$ is constant, which is in turn is much more powerful than Liouville's Theorem that a bounded holomorphic function on $\mathbb{C}$ is constant.

Exercise Deduce Picard's Theorem from Liouville's Theorem and the fact that $\mathbb{D}$ is the universal cover of the thrice-punctured Riemann sphere.

3.1 Counting critical points, and the exceptional set

Before considering the many properties of Julia sets which follow from Montel, we make a brief excursion into topology to count critical points and derive some consequences for *finite* completely invariant sets.

Definition The *valency* of a critical point c of a rational map f is ν_c, where locally near c the map f has the form $z \to kz^{\nu_c}$ (plus higher order terms). In other words the valency is the 'degree of branching' at c.

The following result gives the delayed precise formulation and proof of Proposition 2.3.

Proposition 3.5 (Riemann-Hurwitz Formula) *If f is a rational map of degree d, then*

$$\sum_c (\nu_c - 1) = 2d - 2$$

where the sum is taken over all critical points of f.

Proof Triangulate the target copy of $\hat{\mathbb{C}}$ in such a way that the critical values of f are all vertices, and pull this triangulation back, via f, to a triangulation of the source copy of $\hat{\mathbb{C}}$. The Euler characteristic of $\hat{\mathbb{C}}$ (number of triangles minus number of edges plus number of vertices) is 2. Apart from at critical points, f is a d to one map and so we obtain the equation

$$2d - \sum_c (\nu_c - 1) = 2$$

and thus

$$\sum_c (\nu_c - 1) = 2d - 2$$

QED

Corollary 3.6 *Let f be a rational map with deg$(f) \geq 2$, and suppose E is a finite completely invariant subset of $\hat{\mathbb{C}}$. Then E contains at most 2 points.*

Proof Suppose E contains k points. Then f must permute these points (since every surjection of a finite set to itself is a bijection) and hence for some q the iterate $f^q = g$ is the identity on E. Suppose g has degree d. Each point $z \in E$ must be a critical point of g, of *valency* d, else $g^{-1}(z)$ would contain points other than z. Hence by Proposition 3.5

$$k(d-1) \leq 2d - 2$$

and therefore $k \leq 2$. QED

Definition The *exceptional set* $E(f)$ of a rational map is the union of all finite completely invariant sets. Corollary 3.6 says $|E(f)| \leq 2$. Note that if $|E(f)| = 1$ then f is conjugate to a polynomial (just conjugate by a Möbius transformation sending the exceptional point to ∞), and if $|E(f)| = 2$ then f is conjugate to some $z \to z^d$, with d a positive or negative integer (just send the two exceptional points to ∞ and 0).

3.2 Properties of Julia sets

For a rational map of degree greater or equal to two we have the following:

1. $J(f) \neq \emptyset$.

Proof. Let f be a rational map of degree $d \geq 2$. Then f^n has degree d^n (this can be proved various ways: if you know about homology groups it follows from the fact that $f_* : H_2(S^2) \to H_2(S^2)$ is the homorphism

$\times d : \mathbb{Z} \to \mathbb{Z}$). If $\{f^n\}_{n\geq 0}$ form a normal family on the whole of $\hat{\mathbb{C}}$ then some subfamily $\{f^{n_j}\}_{j\geq 1}$ converges locally uniformly to a function g, and since $\hat{\mathbb{C}}$ is compact $\exists J$ such that $\forall j > J$ and all $z \in \hat{\mathbb{C}}$ we have $d(f^{n_j}(z), g(z)) < \pi/2$ in the spherical metric. But then $\forall j, k > J$ we have $d(f^{n_j}(z), f^{n_k}(z)) < \pi$, and hence that f^{n_j} is homotopic to f^{n_k} (by the 'straight line homotopy' along the shortest great circle arc between $f^{n_j}(z)$ and $f^{n_k}(z)$). Hence $\forall j, k > J$ we have $deg(f^{n_j}) = deg(f^{n_k})$ i.e. $d^{n_j} = d^{n_k}$, contradicting $n_j \to \infty$.

2. $J(f)$ is infinite.

Proof. By Corollary 3.6 the only possibilities for finite completely invariant sets are (up to conjugacy) the set $\{\infty\}$ (for a polynomial) or $\{\infty, 0\}$ (for a map $z \to z^d$). But in both cases these exceptional sets are contained in the Fatou set.

3. $J(f)$ is the smallest completely invariant closed set containing at least three points.

Proof. The complement of a completely invariant closed set containing at least three points is an open completely invariant set omitting at least three points, hence contained in the Fatou set, by Montel's Theorem.

4. $J(f)$ is perfect, that is, every point of $J(f)$ is an accumulation point of $J(f)$.

Proof. For if we let J_0 be the set of accumulation points of J, then J_0 is non-empty (by Property 2), closed (by definition) and completely invariant (using the facts that f is continuous, open and finite-to-one). But J_0 cannot be finite since it would then be exceptional and hence contained in $F(f)$, so $J_0 = J$ by Property 3.

5. $J(f)$ is either the whole of $\hat{\mathbb{C}}$ or it has empty interior.

Proof. Write $S = \hat{\mathbb{C}} - int(J)$. Then S is the union of the Fatou Set F and the boundary ∂J of J, and either S is empty or it is an infinite closed completely invariant set, so containing J (by Property 3).

We remark in connection with Property 5 that there exist examples of rational maps f having $J(f) = \hat{\mathbb{C}}$ (e.g. the example of Lattès (1918): $z \to (z^2+1)^2/4z(z^2-1))$ but that for a *polynomial* map the Fatou set always contains the point ∞ and hence is non-empty.

3.3 Useful results for plotting $J(f)$

Proposition 3.7 *If $deg(f) \geq 2$ and U is any open set meeting $J(f)$, then $\bigcup_{n=0}^{\infty} f^n(U) \supset \hat{\mathbb{C}} - E(f)$.*

Proof If $\bigcup_{n=0}^{\infty} f^n(U)$ misses three or more points of $\hat{\mathbb{C}}$ then f^n are a normal family on U by Montel, contradicting $U \cap J \neq \emptyset$. But if a non-exceptional z lies in $\hat{\mathbb{C}} - \bigcup_{n=0}^{\infty} f^n(U)$ then for some m and n a point of $f^{-m}(z)$ must lie in $f^n(U)$ (since $\bigcup_{m\geq 0} f^{-m}(z)$ is infinite). Hence $z \in f^{m+n}(U)$. Contradiction. QED.

Corollary 3.8 *If z_0 is not in $E(f)$, then $J(f) \subset \overline{\bigcup_{n\geq 0} f^{-n}(z_0)}$.*

Proof Take any $z \in J(f)$ and neighbourhood U of z. By Proposition 3.7 the given point z_0 lies in some $f^n(U)$. Hence $f^{-n}(z_0) \cap U \neq \emptyset$. QED.

This gives us a very simple algorithm for plotting $J(f)$. One just has to start at any (non-exceptional) z_0 whatever and plot all its images under f^{-1}, then all of their images under f^{-1} etc., or alternatively plot $z_0, z_1, z_2, ...$ where each z_{j+1} is a random choice out of the (finite) set of values of $f^{-1}(z_j)$. The resulting set accumulates on the whole of $J(f)$. Even better, if one starts at a point z_0 known to be in $J(f)$ (for example a repelling fixed point) one has $J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(z_0)}$, so that all points plotted are actually in the Julia set, not just accumulating there.

3.4 Julia sets and repelling periodic points

Obviously every repelling periodic point of f lies in the Julia set. However it is also true that every point of the Julia set has a periodic point arbitrarily close to it:

Theorem 3.9 *If $deg(f) \geq 2$ then $J(f)$ is contained in the closure of the set of all periodic points of f.*

Proof Let $z_0 \in J(f)$ and assume z_0 is not a critical value of f (without loss of generality, since there are only a finite set of critical values and $J(f)$ is perfect). Then z_0 has a neighbourhood U on which two distinct branches of f^{-1} are defined. Denote these by $h_1 : U \to U_1$ and $h_2 : U \to U_2$ (where the sets U_1 and U_2 are disjoint).

Suppose (for a contradiction) that U contains no periodic point of f. For each $z \in U$ set

$$g_n(z) = \frac{(f^n z - h_1 z)}{(f^n z - h_2 z)} \frac{(z - h_2 z)}{(z - h_1 z)}$$

Then for each value of n, $g_n(z) \neq 0, 1, \infty$ for $z \in U$ (else f would have a periodic point). So by Montel's Theorem the $\{g_n\}$ form a normal family. It follows (by an exercise in analysis) that the $\{f^n\}$ form a normal family, contradicting the hypothesis that $z_0 \in J(f)$. QED

Comments

1. In fact with a little more work one can establish that $J(f)$ is *equal* to the closure of the set of all *repelling* periodic points of f: this follows from Theorem 3.9 together with the observations that every repelling periodic point of f lies in the Julia set, and the result (of Fatou) that there are only finitely many non-repelling periodic orbits. What Fatou showed was that non-repelling periodic points must either have critical points in their basins of attraction or on the boundaries of their basins (we shall investigate these basins shortly). Shishikura (1987) improved Fatou's result to show that a degree d rational map has at most $2d - 2$ non-repelling periodic orbits.

2. Earlier we observed that for the map $z \to z^2$ the Julia set (the unit circle) is the closure of the set of repelling periodic points. Theorem 3.9 shows that this example typifies the general case.

3.5 The Julia set of $q_c : z \to z^2 + c$ for $|c|$ large

Lemma 3.10 *Let $|c| > 1$. Then $|q_c(z)| > |z|(|c| - 1)$ whenever $|z| \geq |c|$.*

Proof $|q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \geq |z|(|c| - 1)$. QED

Thus if $|c| > 2$ the orbit $z_n = q_c^n(0)$ converges to ∞ as $n \to \infty$, since $z_1 = c$ and $|z_n| \geq |z_{n-1}|(|c| - 1)$.

Definition A *Cantor set* is a topological space homeomorphic to the space $C = \{0, 1\}^{\mathbb{N}}$ of all infinite sequences of 0's and 1's, equipped with the product topology (that is, two sequences are close if and only if they have the identical initial segments, and the longer these identical segments, the closer the points). Recall that every perfect totally disconnected compact subset of $\mathbb{R}^n$ is homeomorphic to C (an example is the Cantor set obtained by removing the open interval $(1/3, 2/3)$ from the closed unit interval on the real line, then the 'middle thirds' $(1/9, 2/9) \cup (7/9, 8/9)$ of the remaining intervals, then the middle thirds of the remaining intervals and so on).

Proposition 3.11 *For $|c|$ sufficiently large, $J(q_c)$ is homeomorphic to the Cantor set C, and the action of q_c on $J(q_c)$ is conjugate to that of the shift σ on C.*

Proof Let γ_0 be the circle $|z| = |c|$, and let $\gamma_1 = q_c^{-1}(\gamma_0)$. Then, if $|c| > 2$, γ_1 lies inside γ_0 (by Lemma 3.10) and γ_1 is a *lemniscate* (since 0 is the only critical point of q_c on $\mathbb{C}$). $q_c^{-1}(\gamma_1)$ now consists of a lemniscate inside each lobe of γ_1, and so on (Figure **??**).

Let D be any disc containing γ_1 and contained in γ_0. Label the the two discs making up $q_c^{-1}(D)$ as D_0 and D_1, and label the components of $q_c^{-2}(D)$ by

$$D_{00} = D_0 \cap q_c^{-1}(D_0) \quad D_{01} = D_0 \cap q_c^{-1}(D_1) \quad D_{10} = D_1 \cap q_c^{-1}(D_0) \quad D_{11} = D_1 \cap q_c^{-1}(D_1)$$

Continue inductively, setting

$$D_{0s} = D_0 \cap q_c^{-1}(D_s) \quad D_{1s} = D_1 \cap q_c^{-1}(D_s)$$

for any finite sequence s of 0's and 1's. Set

$$\Lambda = \bigcap_1^\infty q_c^{-n}(D)$$

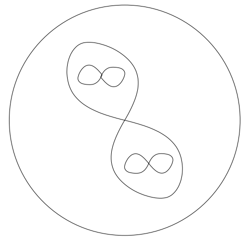

Figure 6: The circle γ_0 and its inverse images (Proposition 3.11).

To show that Λ is a Cantor set we observe that for large $|c|$ both branches of q_c^{-1} contract distances, by a definite factor $k < 1$, on both D_0 and D_1 (the details are in Comment 1 below). Λ therefore consists of points D_s, each labelled by an infinite sequence s of 0's and 1's, and the action of q_c on Λ is conjugate to the action of the shift σ on these sequences. Since Λ is a closed completely invariant set it contains $J(q_c)$; moreover since Λ contains a dense orbit (just write down an infinite sequence of 0's and 1's containing *all* finite sequences) it is a minimal closed completely invariant set and is therefore equal to $J(q_c)$. QED

Comments

1. To show that q_c^{-1} contracts on D_0 and D_1 we must show that q_c expands on their inverse images. But q_c contracts at a point z if and only if $|2z| < 1$, which is to say if and only if $|z| < 1/2$. So it will suffice to show that q_c maps the disc having centre 0, radius 1/2, to the region outside the lemniscate γ_1 (and hence outside both D_0 and D_1). But q_c maps this disc to the disc which has centre c and radius 1/4, and the largest modulus of any point of γ_1 is $|\sqrt{-2c}|$ (exercise). It follows that if c is sufficiently large (for example $|c| > 3$) the image disc lies outside γ_1 and so q_c^{-1} contracts on D_0 and D_1 as required.

2. In fact Proposition 3.11 holds whenever $q_c^n(0) \to \infty$, not just for 'large' $|c|$, but the proof requires a little more work to show that the D_s (s an infinite sequence of 0's and 1's) are points. This is best done by an argument appealing to 'moduli of annuli' (Grötzsch's inequality) or by a normal families argument applied to the branches of q_c^{-1} (see Beardon).

3. For maps in the family q_c, the Julia set $J(q_c)$ is either a Cantor set or else is connected. For if the orbit $q_c^n(0)$ does not tend to ∞ one can show that the basin of attraction of ∞ is a (topological) disc, with boundary a minimal closed completely invariant non-empty set, in other words $J(q_c)$.

4 Fatou components and linearisation theorems

4.1 Counting components

Proposition 4.1 *The Fatou set of a rational map f of degree at least two contains at most two completely invariant simply-connected components.*

Proof Any such component is homeomorphic to a disc D, and the restriction of f to D is a branched covering of degree d. Since D has Euler characteristic 1 we deduce that f has $d-1$ critical points on D (counted with multiplicity). But f has only $2d-2$ critical points. QED.

Example The Fatou set for $z \to z^2$ has exactly two such components.

Omitting the words 'completely invariant simply-connected' and just counting components, we have:

Proposition 4.2 *If $F(f)$ has more than two components, it has infinitely many components.*

Proof If $F(f)$ has only finitely many components, $D_1, ...D_k$, they must be permuted by f (since each component has image a component and inverse image a union of components). Hence there exists an m such that $g = f^m$ maps each D_j to itself. But $F(g) = F(f)$ (from the definition of a normal family) and the D_j are completely invariant for g. To apply Proposition 4.1 and complete the proof it remains to show that the D_j are simply-connected. But each D_j has boundary ∂D_j closed and completely invariant under g, and hence $\partial D_j = J(f)$. It follows that

$$\hat{\mathbb{C}} - \bar{D}_1 = \hat{\mathbb{C}} - (J(f) \cup D_1) = F(f) - D_1 = D_2 \cup ... \cup D_k$$

Hence $D_2, ...D_k$ are the components of the complement of the connected set D_1 and are therefore simply-connected. Similarly D_1 is simply-connected. QED

Examples

(i) $z \to z^2 - 1$. The basin of infinity is a completely invariant component. The components containing 0 and -1 form a periodic 2-cycle. All other components are pre-periodic (fall onto the period two cycle after a finite number of steps).

(ii) $z \to z^2 + c$ with $|c|$ large. Here $F(f)$ has a single component, the complement in $\hat{\mathbb{C}}$ of a Cantor set (but note that this component is multi-connected).

A key theorem concerning the components of $F(f)$ is:

Theorem 4.3 (Sullivan's 'No Wandering Domains Theorem' 1985) *Every component of $F(f)$ is either periodic or preperiodic*

For a proof see Sullivan (Annals 1985), or Appendix F of Milnor's book. The basic idea is that if there were a wandering domain then it would be possible to construct an infinite-dimensional family of perturbations of f, all of them rational and topologically conjugate to f, but this is impossible since f is determined by a finite set of data (as already remarked earlier). The key ingredient is the quasi-conformal deformation theory developed by Ahlfors and Bers, in particular the 'Measurable Riemann Mapping Theorem', which we may consider later in this course. The original conjecture that f could not have wandering domains was made by Fatou. Note that Theorem 4.3 is a result about *rational* maps: the Fatou set of a *transcendental* map $\mathbb{C} \to \mathbb{C}$ can have wandering components (these are known as Baker domains).

The *basin* of an attractive fixed point z_0 is the set $\{z : \lim_{n\to\infty} f^n(z) = z_0\}$ and the *immediate basin* is the component of this set containing z_0. There are similar definitions for an attracting period n cycle: here the immediate basin is the set of components of the basin containing points of the cycle.

Theorem 4.4 *The immediate basin of an attractive periodic point (for a rational map f of degree at least two) contains a critical point.*

Proof Without loss of generality we suppose z_0 to be an attracting *fixed point.* If z_0 is superattracting, the result is obvious. if z_0 is attracting but not superattracting then there is a neighbourhood U of z_0 such that

$f(U) \subset U$ and $f|_U$ is injective. Let $V = f(U)$ and consider the branch of f^{-1} sending V to U. If f has no *critical value* in U, this branch can be extended to the whole of U and hence f^{-2} has a well-defined branch on V. Repeat. If some $f^{-n}(V)$ contains a critical value then the basin contains a critical point. but if not, then $\{f^{-n}\}_{n>0}$ are all defined on V and have images in the the immediate basin. But then they would form an equicontinuous family (by Montel) and this is impossible since z_0 is a *repelling* fixed point for f^{-1}. QED

Corollary 4.5 *If f has degree d then it has at most $2d-2$ attracting or superattracting cycles.*

Shishikura (1987) improved this bound to 'at most $2d-2$ non-repelling cycles'.

4.2 Linearisation Theorems

Dynamics of f near a fixed or periodic point

In the neighbourhood of a fixed point, which without loss of generality we take to be 0, $f(z) = \lambda z + O(z^2)$ (Taylor series), where λ is the multiplier at the fixed point. We say that f is *linearisable* if there is a neighbourhood U on which f is *conjugate* to $z \to \lambda z$ (by a complex analytic conjugacy).

Theorem 4.6 (Koenigs' Linearization Theorem 1884) *If $\lambda \neq 0$ and $|\lambda| \neq 1$ then f is linearizable*

Proof Assume first that $0 < |\lambda| < 1$. Set

$$h_n(z) = \frac{1}{\lambda^n} f^n(z)$$

Then, by construction $h_n f(z) = \lambda h_{n+1}(z)$, and it suffices to show that the $\{h_n\}$ converge locally uniformly to a function h, since then $hf = \lambda h$. See the example below for a sketch proof in the case of a particular example, and Milnor (Theorem 8.2) for the general case, which proceeds along the same lines.

For the case $1 < |\lambda| < \infty$ one can proceed in exactly the same fashion, but with f^{-1} in place of f. QED.

Example

$f(z) = \lambda z + z^2$ (where $|\lambda| < 1$). Here the orbit of any initial point z_0 is

$z_1 = f(z_0) = \lambda z_0(1 + z_0/\lambda)$

$z_2 = f(z_1) = \lambda z_1(1 + z_1/\lambda) = \lambda^2 z_0(1 + z_0/\lambda)(1 + z_1/\lambda)$

...

$z_n = \lambda^n z_0(1 + z_0/\lambda)(1 + z_1/\lambda)...(1 + z_{n-1}/\lambda)$

Thus $h_n(z_0) = z_0(1 + z_0/\lambda)(1 + z_1/\lambda)...(1 + z_{n-1}/\lambda)$ where $\{z_n\}$ is the orbit of z_0. As n tends to infinity, z_n tends to 0, and $\{h_n\}$ converge locally uniformly to

$$h(z_0) = z_0 \prod_0^\infty (1 + \frac{z_n}{\lambda})$$

Observe that we have used the dynamics to construct an explicit conjugacy: essentially we have followed an orbit in to very close to the attracting fixed point, and then used the fact that very close the fixed point the map f is very close to $z \to \lambda z$. One can also construct the coefficients of h recursively, directly from the functional equation $hf(z) = \lambda h(z)$, but the dynamical motivation is then no longer so apparent.

Theorem 4.7 (Böttcher 1904) *If $f(z) = z^k + O(z^{k+1})$ ($k \geq 2$ integer) then f is conjugate to $z \to z^k$ on a neighbourhood of* 0.

Proof Analogously to 4.6, we set $h_n(z) = (f^n(z))^{1/k^n}$. Then $h_n f(z) = (h_{n+1}(z))^k$ and the $\{h_n\}$ converge locally uniformly to a function h conjugating f to $z \to z^k$. QED

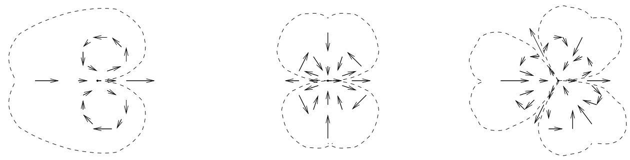

Figure 7: Dynamics of $z \to z + z^{n+1}$ for $n = 1$, $n = 2$ and $n = 3$.

The proof above is only a sketch. See Milnor (Theorem 9.1) for details. The right choice of branch of k^nth root in the definition of h_n is important, but rather than fill in the details in general, we consider an example, one that will also be useful later.

Example Consider $f : z \to z^2 + c$ near the fixed point ∞.

Write this map as $z \to z^2(1 + c/z^2)$.

$z_1 = f(z_0) = z_0^2(1 + c/z_0^2)$

$z_2 = f(z_1) = z_1^2(1 + c/z_1^2) = z_0^4(1 + c/z_0^2)^2(1 + c/z_1^2)$

...

$z_n = z_{n-1}^2(1 + c/z_{n-1}^2) = z_0^{2^n}(1 + c/z_0^2)^{2^{n-1}}(1 + c/z_1^2)^{2^{n-2}}...(1 + c/z_{n-1}^2)$

So $h_n(z_0) = z_0(1 + c/z_0^2)^{1/2}(1 + c/z_1^2)^{1/4}...(1 + c/z_{n-1}^2)^{1/2^n}$ where the choice of each root is the obvious one coming from the binomial expansion. As n tends to ∞ the z_n tend to ∞ (since z_0 is outside the filled Julia set). Thus the h_n converge (locally uniformly) to

$$h(z_0) = z_0 \prod_0^\infty (1 + \frac{c}{z_n^2})^{1/2^{n+1}}$$

Once again one could compute explicit formulae for the coefficients of h using recursion relations based on the functional equation, but they are far less revealing than the dynamical approach above.

We shall come back to this example when we look at the Mandelbrot set later. Meanwhile, what can be said about linearisability near a *neutral* fixed point ?

Suppose $f(z) = \lambda z + O(z^2)$, with $|\lambda| = 1$.

Case 1: $\lambda = e^{2\pi i p/q}$ (in this case $z = 0$ is called a *parabolic* fixed point).

Example $f(z) = z + z^{n+1}$

See Figure **??** for ths example in the cases $n = 1$, $n = 2$ and $n = 3$. The 'attracting petals' bounded by dashed lines are mapped into themselves and each initial point in a petal has orbit which eventually converges to to the fixed point along a direction tangent to the mid-line of the petal. The Julia set (not marked) heads off from the fixed point in directions tangent to the repelling axes (between the petals).

A rational map f is not linearizable around a parabolic fixed point (unless $f(z) = \lambda z$), since $f^q \neq identity$. But by analysing the local power series expansion of $f(z)$ it can be shown that the parabolic point itself lies in the Julia set and its the basin of attraction lies in the Fatou set (See Milnor, Lemma 10.5). It can easily be proved (via Montel) that this basin of attraction must contain a critical point. Similar considerations apply to a parabolic cycle.

The local dynamics around a parabolic fixed point (or cycle) has a very particular topological dynamics, that of a *Leau-Fatou flower*, with 'attracting petals' contained within the Fatou set, as illustrated in the examples above. For $\lambda = e^{2\pi 1 p/q}$ this flower has kq petals, where $k \geq 1$ (see, for example Milnor, Theorem 10.7). The study of holomorphic germs around parabolic points and cycles contains deep and interesting results: Chapter 10 of Milnor's book is an excellent starting point to learn more about this.

Case 2: $\lambda = e^{2\pi i\alpha}$ with α irrational.

Here it all depend on 'how irrational α is'. Write α as a continued fraction

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0, a_1, a_2, \ldots]$$

and let p_n/q_n (in lowest terms) be the value of its nth truncation $[a_0, a_1, \ldots, a_n]$.

For example the golden mean $[0, 1, 1, 1, 1, \ldots]$ has $p_1/q_1 = 1/1, p_2/q_2 = 1/2, p_3/q_3 = 2/3, p_4/q_4 = 3/5, \ldots$.

Definition α satisfies the *Brjuno condition* if and only if

$$\sum_1^\infty \frac{log(q_{n+1})}{q_n} < \infty$$

We write $\mathcal{B}$ for the set of real numbers satisfing the Brjuno condition.

Theorem 4.9 (Brjuno, 1965) $\alpha \in \mathcal{B} \Rightarrow$ *all complex analytic maps* $z \to e^{2\pi i\alpha}z + O(z^2)$ *are linearisable.*

Theorem 4.10 (Yoccoz, 1988) $\alpha \notin \mathcal{B} \Rightarrow z \to e^{2\pi i\alpha}z + z^2$ *is not linearisable.*

When a linearisation exists its domain is known as a *Siegel disc.*

Notes

1. Yoccoz's proof of the *necessity* of the Brjuno condition is motivated by ideas of *renormalization.*

2. The Siegel disc around a linearizable irrational neutral fixed point is in the Fatou set $F(f)$. It can be shown the Siegel discs 'use up' critical points in the sense that the boundary of a Siegel disc necessarily lies in the accumulation set of the forward orbit of some critical point.

3. The irrational neutral points which are not linearizable are known as *Cremer points* (after Cremer 1928). They lie in $J(f)$ and the dynamics around them is complicated. In the 1990s Perez-Marco introduced invariant structures he called 'hedgehogs' and showed they they exist at all Cremer points. These are the subject of continuing research.

4.3 The classification of types of Fatou component

Sullivan's proof of the 'No Wandering Domains Theorem' has the consequence that for a polynomial the only possible components of a Fatou set are components of the basin of:

1. a superattracting periodic orbit;

2. an attracting periodic orbit;

3. a rational neutral periodic orbit;

4. a periodic cycle of Siegel discs.

There is one other type that can ocur for rational f (but not polynomial f), components of the basin of:

5. a periodic cycle of *Herman rings.* (A Herman ring is an annulus with dynamics conjugate to an irrational rotation.)

For a proof of this classification see for example Milnor's book (Chapter 16) or the original paper of Sullivan in 1985.

These are the 5 types of 'regular behaviour' of a rational map. To completely understand rational maps we have to understand how they fit together with each other, and with the behaviour on the complement of the regular domain, the Julia set. As we shall see, there are still unanswered questions even in the simplest case, that of quadratic maps $z \to z^2 + c$.