5 Hyperbolic 3-space and Kleinian groups

Definition $\mathcal{H}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$

Just as in the two-dimensional case we may define an infinitesimal metric:

$$ds = \frac{1}{x_3}((dx_1)^2 + (dx_2)^2 + (dx_3)^2)^{1/2}$$

With this metric $\mathcal{H}^3_+$ becomes the *upper half-space model of hyperbolic* 3*-space*. The geodesics are the semicircles in $\mathcal{H}^3$ orthogonal to the plane $x_3 = 0$.

Now think of the plane $x_3 = 0$ in $\mathbb{R}^3$ as the complex plane $\mathbb{C}$ ($(x_1, x_2, 0) \leftrightarrow x_1 + ix_2$), add the point '$\infty$', and think of $\hat{\mathbb{C}}$ as the *boundary* of $\mathcal{H}^3_+$. Every fractional linear map

$$\alpha : z \to \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{C}, ad - bc \neq 0)$$

mapping $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, has an extension to an isometry from $\mathcal{H}^3_+$ to $\mathcal{H}^3_+$. One way to see this is to break down α into a composition of maps of the form

$$(i) \;\; z \to z + \lambda \quad (\lambda \in \mathbb{C}) \qquad (ii) \;\; z \to \lambda z \quad (\lambda \in \mathbb{C}) \qquad (iii) \;\; z \to -1/z$$

We extend these as follows on $\mathcal{H}^3_+$ (where z denotes $x_1 + ix_2$):

$$(i) \;\; (z, x_3) \to (z + \lambda, x_3) \qquad (ii) \;\; (z, x_3) \to (\lambda z, |\lambda| x_3) \qquad (iii) \;\; (z, x_3) \to \left(\frac{-\bar{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2} \right)$$

The expressions above come from decomposing the action on $\hat{\mathbb{C}}$ of each of the elements of $PSL(2, \mathbb{C})$ in question into two *inversions* (reflections) in circles in $\hat{\mathbb{C}}$. Each such inversion has a unique extension to $\mathcal{H}^3_+$ as an inversion in the hemisphere spanned by the circle and composing appropriate pairs of inversions gives us these formulae. It is now an exercise along the lines of Proposition 2.12 to show that $PSL(2, \mathbb{C})$ preserves the metric ds on $\mathcal{H}^3$ and another exercise, along the lines of Proposition 2.13 to show that the geodesics are the arcs of semicircles as claimed. Moreover every isometry of $\mathcal{H}^3$ can be seen to be the extension of a conformal map of $\hat{\mathbb{C}}$ to itself, since it must send hemispheres orthogonal to $\hat{\mathbb{C}}$ to hemispheres orthogonal to $\hat{\mathbb{C}}$, hence circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$. Thus all orientation-preserving isometries of $\mathcal{H}^3$ are given by elements of $PSL(2, \mathbb{C})$ acting as above, and all orientation-reversing isometries are extensions of anti-holomorphic Möbius transformations of $\hat{\mathbb{C}}$.

Comments

1. The fact that the orientation-preserving isometry group of $\mathcal{H}^3_+$ is $PSL(2, \mathbb{C})$ was first observed by Poincaré.

2. To verify that the extension of the action of $PSL(2, \mathbb{C})$ from $\hat{\mathbb{C}}$ to $\mathcal{H}^3_+$ is well-defined we should check that when we decompose an element of $PSL(2, \mathbb{C})$ into a product in different ways we get the same extension to $\mathcal{H}^3_+$. We can avoid this problem by writing down a single formula for the action of an element of $PSL(2, \mathbb{C})$ in terms of *quaternions*. (Regard $\mathbb{R}^3_+$ as quaternions of the form $x + yi + tj (+0k)$ with $t > 0$: see Exercise Sheet 3.)

3. In practice we may do many of our computations in $\mathcal{H}^3_+$ by taking a hyperplane 'slice' that looks like $\mathcal{H}^2_+$: given any two points P and Q in $\mathcal{H}^3_+$, the plane through these points orthogonal to the boundary $\hat{\mathbb{C}}$ of upper half-space is a copy of $\mathcal{H}^2$, and so $d(P, Q) = |\ln(P, Q; A, B)|$ where A and B are the endpoints of the semicircle through P and Q orthogonal to $\hat{\mathbb{C}}$.

4. The *disc model* for hyperbolic three-space is the interior $\mathbb{D}^3$ of the unit disc in Euclidean three-space $\mathbb{R}^3$, equipped with the metric

$$ds = \frac{((dx_1)^2 + (dx_2)^2 + (dx_3)^2))^{1/2}}{1 - r^2}$$

(where $r^2 = x_1^2 + x_2^2 + x_3^2$). Geodesics are arcs of circles orthogonal to the boundary sphere S^2 of the disc.

5. One can construct higher dimensional hyperbolic spaces $\mathcal{H}_+^n$ in the analagous way. In each case the *conformal* transformations of the boundary extend uniquely to give the *isometries* of the interior.

5.1 Types of isometries of hyperbolic 3-space

Non-identity elements $\alpha \in PSL(2, \mathbb{C})$ are of four types.

Definition α is said to be

elliptic $\Leftrightarrow$ α fixes some geodesic in $\mathcal{H}_+^3$ pointwise;

parabolic $\Leftrightarrow$ α has a single fixed point in $\hat{\mathbb{C}}$;

hyperbolic $\Leftrightarrow$ α has two fixed points in $\hat{\mathbb{C}}$, no fixed points in $\mathcal{H}_+^3$, and every hyperplane in $\mathcal{H}_+^3$ which contains the geodesic joining the two fixed points in $\hat{\mathbb{C}}$ is invariant (mapped to itself) under α;

loxodromic $\Leftrightarrow$ α has two fixed points in $\hat{\mathbb{C}}$, no fixed points in $\mathcal{H}_+^3$, and no invariant hyperplane in $\mathcal{H}_+^3$.

Note The distinction between *hyperbolic* and *loxodromic* is not always made: some authors use either word for an isometry having two fixed points in $\hat{\mathbb{C}}$ and none in $\mathcal{H}_+^3$.

Lemma 5.1 *α is elliptic/parabolic/hyperbolic/loxodromic*

$\Leftrightarrow (tr(\alpha))^2 \in [0, 4) \subset \mathbb{R}^{\geq 0}\ /\ = 4\ /\ \in \mathbb{R}^{\geq 0} - [0, 4]\ /\ \in \mathbb{C} - \mathbb{R}^{\geq 0}$ *(where α has been normalised to have det $= 1$).*

Proof

If α has two fixed points in $\hat{\mathbb{C}}$ we may assume (after conjugating α by an appropriate Möbius transformation) they are at 0 and ∞ and that α has the form $z \to \lambda z$ (and $tr(\alpha) = \lambda^{1/2} + \lambda^{-1/2}$).

Case 1: $|\lambda| = 1$, say $\lambda = e^{i\theta}$. Then on $\hat{\mathbb{C}}$ α is a rotation about 0 through an angle θ, and fixes the x_3-axis in $\mathcal{H}_+^3$ pointwise. As a matrix, normalised to determinant 1,

$$\alpha = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

and so $(tr(\alpha))^2 = 4\cos^2(\theta/2) \in [0, 4)$.

Case 2: $|\lambda| \neq 1$. then α acts on the x_3-axis in $\mathcal{H}_+^3$ as multiplication by $|\lambda|$. Writing $\lambda = |\lambda|e^{i\theta}$ we have

$$\alpha = \begin{pmatrix} |\lambda|^{1/2}e^{i\theta/2} & 0 \\ 0 & |\lambda|^{-1/2}e^{-i\theta/2} \end{pmatrix}$$

so $(tr(\alpha))^2 \in \mathbb{C} - [0, 4]$. Now if λ is real (i.e. $\theta = 0$ or π) α is hyperbolic and $(tr(\alpha))^2 \in \mathbb{R}^{\geq 0} - [0, 4]$ and if λ is not real, α is loxodromic and $(tr(\alpha))^2 \in \mathbb{C} - \mathbb{R}^{\geq 0}$.

Finally if α has a single fixed point in $\hat{\mathbb{C}}$ then we can place this fixed point at ∞ (by conjugating α if necessary) in which case α has the form $z \to z + \lambda$ (indeed we may even conjugate it to $z \to z + 1$). Then α is parabolic and $(tr(\alpha))^2 = 4$. QED.

Dynamics of Möbius transformations on $\mathcal{H}_+^3 \cup \hat{\mathbb{C}}$

In the first example in Figure 8 the fixed points $0, \infty$ on $\hat{\mathbb{C}}$ are *neutral*. For $z \to e^{i\theta}z$ with θ real, all orbits on $\mathcal{H}_+^3$ have finite period if θ is a rational multiple of π, and densely fill circles around the x_3 axis if not.

In the second example all orbits in $\mathcal{H}_+^3$ head away from a repelling fixed point 0 and towards an attracting fixed point ∞, spiralling around the x_3 axis as they go. We have this behaviour in general for $z \to ke^{i\theta}z$ (k real > 1) but the nature of the spiralling depends on θ: in particular if $\theta = 0$ or π each orbit remains in a hyperplane.

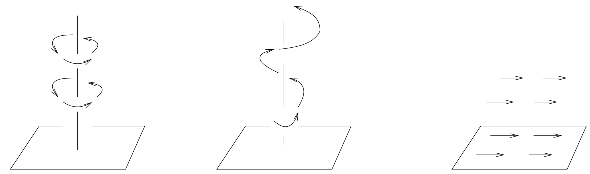

Figure 8: Dynamics of (i) $z \to e^{2\pi i/3} z$ (ii) $z \to 2e^{2\pi i/3} z$ (iii) $z \to z + 1$

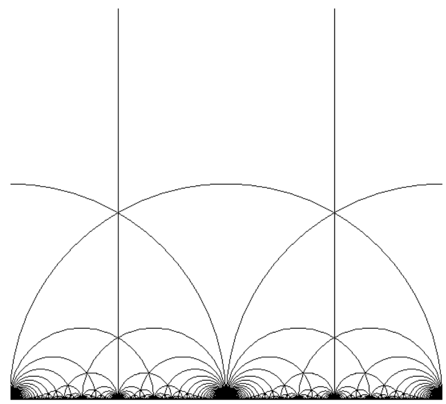

Figure 9: The modular group action on the upper half-plane

In the third example the (unique) fixed point ∞ is neutral (multiplier 1) and all orbits on $\mathcal{H}^3_+$ head towards the fixed point under both forward and backward time. Any parabolic map α will have this behaviour.

5.2 The ordinary set of a Kleinian group

Definition A *Kleinian group* is a *discrete* subgroup $G < PSL(2, \mathbb{C})$.

Thus for a subgroup $G < PSL(2, \mathbb{C})$ to be called Kleinian we require that there be no sequence $\{g_n\}$ of distinct elements of G tending to a limit $g \in PSL(2, \mathbb{C})$. Here the topology on $PSL(2, \mathbb{C})$ is that induced by the norm

$$\left|\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|\right| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$$

on $SL(2, \mathbb{C})$ (so that two elements of $PSL(2, \mathbb{C})$ are close together if and only if they are representable by $A_1, A_2 \in SL(2, \mathbb{C})$ with $||A_2 - A_1||$ small).

Note If G is discrete then for any $N > 0$ the number of elements of G having norm $\leq N$ is *finite*, since every infinite sequence with bounded norm has a convergent subsequence. Hence every discrete G is *countable*.

Definition The action of G is *discontinuous* at $z \in \hat{\mathbb{C}}$ if there exists a neighbourhood U of z such that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$.

Example (See Week 3 Exercises) $G = PSL(2, \mathbb{Z})$ acts discontinuously on $\hat{\mathbb{C}} - \hat{\mathbb{R}}$. For z in the region $\Delta = \{z : |z| \leq 1, Re(z) \leq 1/2, Im(z) > 0\}$ (Figure 9) each $z \neq i, \pm 1/2 + i\sqrt{3}/2$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all non-identity $g \in G$, the point $z = i$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I, S\}$ where $S : z \to -1/z$, and the point $z = -1/2 + i\sqrt{3}/2$ has a neighbourhood U such that $g(U) \cap U = \emptyset$ for all $g \in G - \{I, ST, (ST)^2\}$ where $ST : z \to -1/(z + 1)$, etc.

Definition The set of all $z \in \hat{\mathbb{C}}$ at which the action of G is discontinuous is called the *ordinary* (or *discontinuity* or *regular*) set $\Omega(G)$.

Comments

1. It follows at once from the definition that $\Omega(G)$ is *open* and *G-invariant*.

2. In the example in figure 9 observe that the origin 0 is not in $\Omega(G)$, since any U containing 0 has $g(U) \cap U \neq \emptyset$ for all

$$g = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

with n sufficiently large. In fact in this example $\Omega(G) = \hat{\mathbb{C}} - \hat{\mathbb{R}}$.

5.3 The action of a Kleinian group on $\mathcal{H}^3_+$

We next consider the action of a subgroup $G < PSL(2, \mathbb{C})$ on $\mathcal{H}^3_+$ rather than just on its boundary $\hat{\mathbb{C}}$.

Theorem 5.1 *A subgroup $G < PSL(2, \mathbb{C})$ is discrete if and only if it acts discontinuously on $\mathcal{H}^3_+$.*

Proof. If G is not discrete there exists $\{g_n\} \in G$ with limit $g \in PSL(2, \mathbb{C})$. So for all $x \in \mathcal{H}^3_+$, $g_m^{-1} g_n(x) \to x$ as $m, n \to \infty$. Thus for any $x \in \mathcal{H}^3_+$ and neighbourhood U of x, for m and n sufficiently large $g_m^{-1} g_n(U) \cap U \neq \emptyset$. Hence G does not act discontinuously at x.

Conversely, if G does not act discontinuously at $x \in \mathcal{H}^3_+$, then for any neighbourhood U of x there exist a sequence $\{x_n\} \in U$ and (distinct) $g_n \in G$ such that each $g_n(x_n) \in U$. Take U compact. Then by passing to subsequences we may assume the x_n tend to a point y and the $g_n(x_n)$ tend to a point z (with both y and z in U). Now let k be an isometry of $\mathcal{H}^3_+$ having $k(z) = y$ and let $\{h_n\}$, $\{j_n\}$ be sequences of isometries, both tending to the identity, and having $h_n(y) = x_n$ and $j_n g_n(x_n) = z$ respectively. Consider $f_n = k j_n g_n h_n$. For each n this fixes y (by construction). But the isometries of $\mathcal{H}^3_+$ fixing a common point of $\mathcal{H}^3_+$ are a compact group (the Euclidean rotations, in the Poincaré ball model with the common point the origin). Hence the $\{f_n\}$ have a convergent subsequence. Hence so do the $\{g_n\}$, in other words G is not discrete. QED

5.4 Limit sets of Kleinian groups

One can define the notion of the *limit set* $\Lambda(G)$ of a Kleinian group G, either in terms of its action on $\mathcal{H}^3_+$, or in terms of the action on the boundary $\hat{\mathbb{C}}$ of $\mathcal{H}^3_+$. We shall see later that the two definitions are equivalent.

Definition 1. Let x be any point of $\mathcal{H}^3_+$. Then set

$$\Lambda(x) = \{w \in \hat{\mathbb{C}} : \exists g_n \in G \quad \text{with} \quad g_n(x) \to w \quad \text{as} \quad n \to \infty\}$$

(where convergence is taken in the Euclidean metric on the Poincaré disc model of $\mathcal{H}^3_+$). Note that the $\{g_n(x)\}$ cannot have accumulation points in $\mathcal{H}^3_+$, since G acts discontinuously there. Thus an alternative description of $\Lambda(x)$ is as the accumulation set in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$ of the orbit Gx on $\mathcal{H}^3_+$. This accumulation set is independent of the initial point $x \in \mathcal{H}^3_+$, since if we choose another initial point y the hyperbolic distance from $g(x)$ to $g(y)$ is constant for all g and therefore the *Euclidean* distance from $g(x)$ to $g(y)$ tends to zero as $g(x)$ and $g(y)$ approach the boundary $\hat{\mathbb{C}}$ of the Poincaré disc. We *define* $\Lambda(G)$ to be $\Lambda(x)$ for any $x \in \mathcal{H}^3_+$.

Definition 2. Let z be any point of $\hat{\mathbb{C}}$. Set

$$\Lambda(z) = \{w \in \hat{\mathbb{C}} : \exists g_n \in G \quad \text{with} \quad g_n(z) \to w \quad \text{as} \quad n \to \infty\}$$

(where convergence is taken in the spherical metric on $\hat{\mathbb{C}}$). It can be shown that when G is *non-elementary* (see below for definition) $\Lambda(z)$ is independent of $z \in \hat{\mathbb{C}}$. We define $\Lambda(G)$ to be $\Lambda(z)$ for any $z \in \hat{\mathbb{C}}$.

Comments

1. The restriction that G be 'non-elementary' is included in definition 2 in order to exclude just one class of examples where the limit $\Lambda(z)$ depends on z. Consider $G = \{g^n : n \in \mathbb{Z}\}$, where g is loxodromic, with fixed points z_0 and z_1. The limit set by definition 1 is $\Lambda(G) = \{z_0\} \cup \{z_1\}$, but definition 2 gives $\Lambda(z_0) = z_0$, $\Lambda(z_1) = z_1$ (although $\Lambda(z) = \{z_0\} \cup \{z_1\}$ for any other choice of z).

2. We shall adopt definition 2 until we have proved the equivalence of the two notions (later in this section). Meanwhile we remark that the underlying reason that the definitions are equivalent is that to an observer inside $\mathcal{H}^3_+$ an orbit of G of $\mathcal{H}^3_+$ is viewed as accumulating at $\Lambda(G)$ on the 'visual sphere' $\hat{\mathbb{C}}$.

3. A third equivalent definition is that $\Lambda(G)$ consists of the points $z \in \hat{\mathbb{C}}$ where the family $g \in G$ fail to be a normal family (with respect, as always, to the spherical metric). We shall prove also later.

4. It follows at once from definition 2 (or indeed from definition 1) that $\Lambda(G)$ is both *closed* and *G-invariant*.

It is clear from the definitions of $\Omega(G)$ and $\Lambda(G)$ that $\Omega(G) \cap \Lambda(G) = \emptyset$, but we shall prove the stronger statement that $\Lambda(G)$ is the *complement* of $\Omega(G)$ in $\hat{\mathbb{C}}$. First we deal with some special cases.

5.5 Elementary Kleinian groups

Definition A Kleinian group G is called *elementary* if there exists a finite G orbit on either $\mathcal{H}^3_+$ or $\hat{\mathbb{C}}$.

All elementary Kleinian groups G belong to the following three classes. For a proof see for example Beardon's book 'Geometry of Discrete Groups' or Ratcliffe's book 'Foundations of Hyperbolic Manifolds.'

(i) G is conjugate to a finite subgroup of $SO(3)$ acting on the Poincaré disc by rigid rotations fixing the origin (for example the symmetry group of a regular solid). In this case $\Lambda(G) = \emptyset$.

(ii) G is conjugate to a discrete group of Euclidean motions of $\mathbb{C}$ (i.e. fixing $\infty \in \hat{\mathbb{C}}$). (For example the group generated by $z \to z + 1$ and $z \to z + i$). Then $|\Lambda(G)| = 1$.

(iii) G is conjugate to a group in which all elements are of the form $z \to kz$ or $z \to k/z$ for $k \in \mathbb{C}$. Then $|\Lambda(G)| = 2$.

It is not hard to see that if G is Kleinian then $\Lambda(G) = \emptyset \Rightarrow G$ elementary of type (i), $|\Lambda(G)| = 1 \Rightarrow G$ elementary of type (ii), and $|\Lambda(G)| = 2 \Rightarrow G$ elementary of type (iii), so elementary groups are characterised by the size of their limit sets. Indeed

Proposition 5.2 *A Kleinian group G is elementary if and only $|\Lambda(G)| \leq 2$, and non-elementary if and only if $\Lambda(G)$ is infinite.*

Proof. If $\Lambda(G)$ is finite and non-empty then any G orbit in $\Lambda(G)$ is a finite G orbit on $\hat{\mathbb{C}}$ so G is elementary by definition and has $|\Lambda(G)| = 1$ or 2 by the above classification. QED

5.6 Properties of ordinary and limit sets

Theorem 5.3 *Every Kleinian group G acts discontinuously on $\hat{\mathbb{C}} - \Lambda(G)$. Hence $\hat{\mathbb{C}}$ is the disjoint union of $\Omega(G)$ and $\Lambda(G)$.*

Proof. *(Outline.)* For groups G with $|\Lambda(G)| = 0, 1$ the result can be verified by checking the corresponding types of elementary Kleinian groups, so we may assume $|\Lambda(G)| \geq 2$. Now let $C(G)$ be the *convex hull* of $\Lambda(G)$ in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$, i.e. the space obtained by joining every point of $\Lambda(G)$ to every other point of $\Lambda(G)$ by a geodesic in $\mathcal{H}^3_+$ and then 'filling in the interior' to obtain a convex set in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$. (An equivalent definition of $C(G)$ is that it is the space obtained from $\mathcal{H}^3_+$ by 'scooping out' every open half 3-ball bounded by a round 2-disc contained in $\hat{\mathbb{C}} - \Lambda(G)$). The set $C(G)$ is closed and G-invariant, since $\Lambda(G)$ is. There is a uniquely defined retraction map

$$\rho : \mathcal{H}^3_+ \cup (\hat{\mathbb{C}} - \Lambda(G)) \to C(G)$$

sending each point of $\mathcal{H}^3$ to the nearest point of $C(G)$ (in the hyperbolic metric). This map ρ is continuous and commutes with the action of G. Now let z be any point of $\hat{\mathbb{C}} - \Lambda(G)$ and $U \subset \hat{\mathbb{C}} - \Lambda(G)$ be a neighbourhood of z. Then $\rho(U)$ is contained in a neighbourhood V of $\rho(z)$, and by taking U small (in the spherical metric) we can take V as small as we please (in the hyperbolic metric). But now since the action of G is discontinuous (by Theorem 5.1) V meets $g(V)$ for at most finitely many $g \in G$. Hence $g\rho(U)$ meets $\rho(U)$ for at most finitely many $g \in G$, and so $g(U)$ meets U for at most finitely many $g \in G$, in other words $z \in \Omega(G)$. QED

Proposition 5.4 *Let G be a non-elementary Kleinian group. Then any non-empty closed G-invariant subset S of $\hat{\mathbb{C}}$ contains $\Lambda(G)$*

Proof. Let z be any point of S having an infinite orbit under G. Since S is G-invariant it contains the orbit Gz, and since S is closed it contains the accumulation set of Gz. But this accumulation set is $\Lambda(G)$. QED

Corollary 5.5 *Let G be a Kleinian group. Then either $\Lambda(G) = \hat{\mathbb{C}}$ or $\Lambda(G)$ has empty interior.*

Proof. In the elementary case $\Lambda(G)$ has empty interior. In the non-elementary case apply Proposition 5.4 to $\hat{\mathbb{C}} - int\Lambda(G)$. QED

Corollary 5.6 *Let G be a non-elementary Kleinian group. Then $\Lambda(G)$ is the closure of the set of all fixed points of loxodromic and hyperbolic elements of G.*

Proof. If $z \in \hat{\mathbb{C}}$ is a fixed point of a hyperbolic or loxodromic element $g \in G$ then z lies in $\Lambda(G)$ by definition 2. For the converse we remark that the set of fixed points of loxodromic and hyperbolic elements of a non-elementary group is non-empty (by a standard exercise) and is G-invariant since if z is fixed by g, then hz is fixed by hgh^{-1}. The result now follows by Proposition 5.4. QED

Comment. If G has any parabolic elements their fixed points must lie in $\Lambda(G)$, but elliptic elements may have fixed points in either $\Omega(G)$ or $\Lambda(G)$.

Corollary 5.7 *Let G be a non-elementary Kleinian group. Then $\Lambda(G)$ is perfect (and hence, in particular, uncountable).*

Proof. The set of accumulation points of $\Lambda(G)$ is closed and G-invariant. Now apply Proposition 5.4. QED

Corollary 5.8 *Definitions 1 and 2 for the limit set $\Lambda(G)$ of a non-elementary Kleinian group G are equivalent.*

Proof. We show that the limit set as defined by definition 1 has exactly the same characterising property as that specified by Proposition 5.4 for $\Lambda(G)$ (where we used definition 2). Let S be any closed G-invariant subset of $\hat{\mathbb{C}}$ (note that S must be infinite, since G is non-elementary). Then $C(S)$, the convex hull of S in $\mathcal{H}^3_+ \cup \hat{\mathbb{C}}$, is also closed and G-invariant. Take any $x \in C(S) \cap \mathcal{H}^3_+$. Its orbit Gx is contained in $C(S)$ and the accumulation set of this orbit is contained in $C(S) \cap \hat{\mathbb{C}} = S$. Hence S contains the definition 1 limit set of G. QED

5.7 Comparison with Fatou and Julia sets

The results we have proved so far for regular and limit sets for Kleinian groups exhibit a very close analogy with our earlier results on Fatou and Julia sets for rational maps. This raises the question as to whether we can make the *definitions* analogous too. The answer is yes.

Proposition 5.9 *Let G be a Kleinian group. Then $\Omega(G)$ is the largest open subset of $\hat{\mathbb{C}}$ on which the elements of G form an equicontinuous family.*

Proof. Assume G non-elementary (as usual elementary groups can be dealt with on a case by case basis). Then $\Lambda(G)$ contains at least three points (in fact infinitely many) so $\Omega(G)$ is contained in the equicontinuity set by Montel's Theorem. But given any $z \in \Lambda(G)$, by Corollary 5.6 there must be a repelling fixed point of some $g \in G$ arbitrarily close to z, so the family of maps G cannot be equicontinuous at z. QED

We deduce the following two consequences (useful for plotting $\Lambda(G)$).

Theorem 5.10 *Let G be a non-elementary Kleinian group, and U be any open subset of $\hat{\mathbb{C}}$ meeting $\Lambda(G)$. Then*

$$\bigcup_{g \in G} gU = \hat{\mathbb{C}}$$

Proof. The union $\bigcup_{g \in G} gU$ covers all of $\hat{\mathbb{C}}$ except at most two points (else the family G would be equicontinuous on U by Montel's Theorem). But the complement of this union is a finite G-invariant set and therefore empty (since G is non-elementary). QED

The following corollary is immediate.

Corollary 5.11 *Let G be a non-elementary Kleinian group, and U be any open subset of $\hat{\mathbb{C}}$ meeting $\Lambda(G)$. Then*

$$\bigcup_{g \in G} g(U \cap \Lambda(G)) = \Lambda(G)$$

Comments

1. A discrete subgroup of $PSL(2, \mathbb{R})$ is called *Fuchsian.* All our results for Kleinian groups in this chapter have obvious specialisations to the Fuchsian case, with $\mathcal{H}^3_+$ replaced by $\mathcal{H}^2_+$, and $\hat{\mathbb{C}}$ replaced by $\hat{\mathbb{R}}$.

2. 'Sullivan's Dictionary' is a continually evolving correspondence between definitions, conjectures and theorems in the realm of iterated rational maps and definitions, conjectures and theorems in the realm of Kleinian groups. Some entries are obvious, e.g. Julia set $\leftrightarrow$ limit set, but not everything works in exactly the same way in the two areas, for example:

Ahlfors $0 - 1$ **Conjecture**, formulated by Ahlfors in the 1960s and proved by him for *geometrically finite* Kleinian groups, states in its most general form that for any finitely generated Kleinian group G either $\Lambda(G) = \hat{\mathbb{C}}$ or $\Lambda(G)$ has 2-dimensional Lebesgue measure zero. This was finally proved in 2004 as a consequence of work by many authors (see Marden, Theorem 5.6.6).

Fatou's Question. Can the Julia set of a polynomial have positive 2-dimensional Lebesgue measure? This question was finally answered in 2005 by Xavier Buff and Arnaud Chéritat, who proved that there exist quadratic polynomials, $z \to z^2 + c$, with positive area Julia sets. The proof is very technical, but see their paper at the 2010 Intenational Congress of Mathematics in Hyderabad for an overview of their method.

I don't know that the current contents of this dictionary are all written down in one place, but see Chapter 5 of the book by S.Morosawa, Y.Nishimura, M.Taniguchi and T.Ueda for the situation in 2000. More recently Dick Canary gave a talk about the dictionary at Dennis Sullivan's 70th birthday conference at Stony Brook in 2011 and you can find this on the web.