8 Quasiconformal mappings: the Measurable Riemann Mapping Theorem and its applications

8.1 The moduli space and the Teichmüller space of a torus

Given any two Riemann surfaces $\mathcal{S}_1, \mathcal{S}_2$ which are homeomorphic to a sphere, there is *conformal* homeomorphism $\mathcal{S}_1 \to \mathcal{S}_2$. This follows from the Uniformisation Theorem, which tells us that every Riemann surface has universal cover $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. But Riemann surfaces which are homeomorphic to the torus are another matter. Every such surface $\mathcal{S}$ has universal cover $\mathbb{C}$, and the group Γ of covering transformations of $\mathcal{S}$ is a subgroup of $Aut(\mathbb{C})$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Thus Γ is generated by two translations of $\mathbb{C}$ in directions which are linearily independent over $\mathbb{R}$. By conjugating by a scaling and a rotation of $\mathbb{C}$ we may assume that one of the translations is $z \to z+1$ and the other is $z \to z + \lambda$, some $\lambda \in \mathbb{C} - \mathbb{R}$.

Proposition 8.1
Let $\mathcal{S}_1$ be the torus $\mathbb{C}/\Gamma_1$, where Γ_1 is generated by $z \to z+1$ and $z \to z + \lambda_1$, and let $\mathcal{S}_2$ be the torus $\mathbb{C}/\Gamma_2$, where Γ_2 is generated by $z \to z+1$ and $z \to z+\lambda_2$. Then there is a conformal homeomorphism (i.e. an analytic bijection) between $\mathcal{S}_1$ and $\mathcal{S}_2$ if and only if $\lambda_2 = g(\lambda_1)$ for some $g \in PSL(2,\mathbb{Z})$.

Proof First observe that if $\lambda_2 = \lambda_1 + 1$ then $\Gamma_2 = \Gamma_1$ so $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same torus, and if $\lambda_2 = -1/\lambda_1$ then the lattice $\Gamma_2 \subset \mathbb{C}$ is obtained from the lattice Γ_1 by rotating and rescaling $\mathbb{C}$, so $\mathcal{S}_2$ is isomorphic to $\mathcal{S}_1$. Since these two operations generate the action of $PSL(2,\mathbb{Z})$ on λ_1, it follows that if $\lambda_2 = g(\lambda_1)$ for any $g \in PSL(2,\mathbb{Z})$ then $\mathcal{S}_2$ is isomorphic to $\mathcal{S}_1$.

Conversely, if $\mathcal{S}_2$ is isomorphic to $\mathcal{S}_1$ then by the Uniformisation Theorem there must exist an automorphism of $\mathbb{C}$, fixing the origin and conjugating the generators $z \to z+1$ and $z \to z + \lambda_1$ of Γ_1 to a pair of generators of Γ_2, that is to say there must exist $0 \neq \mu \in \mathbb{C}$ such that Γ_2 is the group generated by $z \to \mu$ and $z \to \mu\lambda_1$. Since Γ_2 is also generated by $z \to z+1$ and $z \to z+\lambda_2$ this implies there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ such that

$$\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

and so λ_2 is the image of λ_1 under an element of $PSL(2,\mathbb{Z})$. QED

Thus we get a different complex structure on a topological torus for each different point λ in our fundamental domain Δ for the action of the modular group $PSL(2,\mathbb{Z})$. The complex structures on the torus therefore correspond to the points of the *moduli space*

$$\mathcal{M} = \mathcal{H}^2_+/PSL(2,\mathbb{Z})$$

which is a sphere with a puncture point (corresponding to ∞), a cone point of angle π (corresponding to i) and a cone point of angle $2\pi/3$ (corresponding to $(-1+i\sqrt{3})/2$). Given a Riemann surface of genus 1, we can *mark* it by choosing two homotopy classes of loops which generate the fundamental group. This corresponds in the universal cover to choosing generators of the covering transformation group $\Gamma = \mathbb{Z} \times \mathbb{Z}$. The *marked* complex structures on the torus correspond to the points on the universal cover of $\mathcal{M}$, the *Teichmüller space* $\mathcal{T} = \mathcal{H}^2_+$.

Remark. For a genus g surface $\mathcal{S}_g$, with $g \geq 2$, the Teichmüller space $\mathcal{T}(\mathcal{S}_g)$ is a copy of $\mathbb{R}^{6g-6}$ (one can give explicit coordinates in terms of lengths of certain loops on $\mathcal{S}_g$), and the moduli space is the qutioent of Teichmüller space by the *mapping class group* of $\mathcal{S}_g$.

However one can construct a homeomorphism from a Riemann surface of genus g to any other Riemann surface of the same genus if we weaken the requirement of conformality to a requirement that the homeomorphism should 'send infinitesimal circles to infinitesimal ellipses having bounded ratios of internal to external diameter'. Such homeomorphisms are called *quasiconformal homeomorphisms.*

Example Figure 19 illustrates a quasiconformal homeomorphism which sends the small circles on the left hand torus to the small ellipses on the right hand torus.

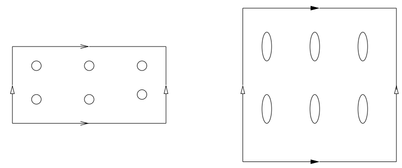

Figure 19: There is a quasiconformal homemorphism between these two tori

8.2 Quasiconformal homeomorphisms and the Measurable Riemann Mapping Theorem

An invertible linear map $\mathbb{R}^2 \to \mathbb{R}^2$ sends a circle centred at the origin to an ellipse centred at the origin, so a C^1-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ sends an infinitesimal circle at each point $x \in \mathbb{R}^2$ to an infinitesimal ellipse at $f(x)$.

Definition A homeomorphism f between open sets in $\mathbb{C}$ is said to be K-*quasiconformal* if it sends infinitesimally small round circles to infinitesimally small ellipses which have ratio of semi-major axis length to semi-minor axis length less than or equal to K. *(Technical point: we do not require that f be C^1, only that f have "distributional derviatives in L^1" . See Milnor, Appendix F.)*

We can write f as a function of z and $\bar{z}$ (if f were conformal it would just be function of z). We can then associate to f the *Beltrami form*:

$$\mu(f) = \frac{\partial f}{\partial \bar{z}} / \frac{\partial f}{\partial z}$$

and it is straightforward to prove that f is K-quasiconformal, with $K = (1+k)/(1-k)$, if and only if $\mu(f)$ is defined almost everywhere and has essential supremum $||\mu||_\infty = k < 1$. (See, for example, Carleson and Gamelin.)

Recall that the *Riemann Mapping Theorem* asserts that if U is a bounded open simply-connected subset of $\mathbb{C}$, then there exists a conformal orientation-preserving homeomorphism $\phi : U \to \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. Clearly ϕ is unique up to post-composition by orientation-preserving conformal homeomorphisms of $\mathbb{D}$, that is to say fractional linear maps which send the unit disc $\mathbb{D}$ to itself.

The *Measurable Riemann Mapping Theorem* asserts the analogous result in the case that in addition to U we are given an assigned *complex dilatation* $\mu(z)$ at every point $z \in U$ (except possibly at points in a set of Lebesgue measure zero) and rather than seeking a conformal homeomorphism ϕ from U to $\mathbb{D}$, what we are looking for is a *quasiconformal* homeomorphism $f : U \to \mathbb{D}$ which has the prescribed dilatation $\mu(z)$ at almost every point $z \in U$. We only require that the assignment $z \to \mu(z)$ be measurable, not that it be continuous.

The Measurable Riemann Mapping Theorem is due to Morrey, Bojarski, Ahlfors and Bers. It has various versions appropriate for different applications. The statement below is that of Théorème 5 in Douady's paper in LMS Lecture Notes Volume 274 'The Mandelbrot set. Theme and Variations' (edited by Tan Lei): it is expressed in terms of functions defined on the whole of $\mathbb{C}$, but can be adapted to suit other situations, for example when the domain of μ is a bounded simply-connected open subset U of $\mathbb{C}$ and we seek a quasiconformal homeomorphism $f : U \to \mathbb{D}$ or indeed when the domain of μ is the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and we seek a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Theorem 8.2 (The Measurable Riemann Mapping Theorem.) *Let μ be any L^∞ function $\mathbb{C} \to \mathbb{C}$ with $||\mu||_\infty = k < 1$. Then there exists an orientation-preserving quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ which has complex dilatation $\mu(f)$ equal to μ almost everywhere on $\mathbb{C}$. This homeomorphism is unique if we require that $f(0) = 0$ and $f(1) = 1$. Furthermore if μ depends analytically (respectively continuously) on a parameter λ then the homeomorphism f also depends analytically (respectively continuously) on λ.*

8.3 1st Application: maps in the same hyperbolic component of the interior of the Mandelbrot set are quasiconformally conjugate

For simplicity consider the component consisting of the interior of the main cardioid M_0. We know that for any $c' \neq c$, both in $int(M_0)$, q_c is not conformally conjugate to $q_{c'}$ since they have different multipliers at the attracting fixed point. However, provided both c and c' are non-zero, $q_{c'}$ is quasiconformally conjugate to q_c. To prove this, consider a small circle γ_0 around the attracting fixed point of q_c (sufficiently small that it does not contain the critical value c). The circle γ_0 and its image $\gamma_1 = q_c(\gamma_0)$ bound an annulus A; the map q_c identifies the outer boundary γ_0 of A with its inner boundary γ_1 and the quotient of A under this identification is a torus $\mathcal{S}$. Similarly for q_c' we obtain an annulus A' and a torus $\mathcal{S}'$. Although $\mathcal{S}'$ is not conformally homeomorphic to $\mathcal{S}$ (since the multipliers of the two quadratic maps at their respective attracting fixed points are different), they are quasiconformally homeomorphic, since this is true for any pair of Riemann surfaces of genus 1. Let h be quasiconformal homeomorphism $\mathcal{S} \to \mathcal{S}'$. The derivative of h sends the field of infinitesimal circles on S to a field of infinitesimal ellipses on $\mathcal{S}'$. We can 'spread' this field of ellipses to the whole of the attracting basin $int(K(q_{c'}))$ of the fixed point of $q_{c'}$ by repeatedly applying $q_{c'}$ and $q_{c'}^{-1}$ to the lift A' of $\mathcal{S}'$. This ellipse field, together with the infinitesimal round circle field on $\hat{\mathbb{C}} - int(K(q_{c'}))$, provides us with an ellipse field on $\hat{\mathbb{C}}$ which is preserved by the map $q_{c'}$. Applying the Measurable Riemann Mapping Theorem to this ellipse field on $\hat{\mathbb{C}}$ yields a quasiconformal conjugacy from $q_{c'} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to a map $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is

(i) holomorphic (because q preserves the field of infinitesimal round circles),

(ii) a polynomial (because $q(\infty) = \infty = q^{-1}(\infty)$), and

(iii) necessarily conformally conjugate to q_c (because by construction q has the correct multiplier at its attracting fixed point).

We can apply a similar argument to any hyperbolic component of the interior of the Mandelbrot set M, replacing 'attracting fixed point' by 'attracting periodic cycle' and 'q_c' by '$(q_c)^n$'. But note that each hyperbolic component of $int(M)$ contains one special value c_0 where the attracting cycle is superattracting (i.e. the critical point 0 is periodic), and that q_{c_0} is not even topologically conjugate to the other q_c's: nevertheless the Julia set $J(q_{c_0})$ for this 'postcritically finite' map q_{c_0} is still quasiconformally homeomorphic to the other $J(q_c)$'s, a fact that can be proved by remembering that the Julia set is the closure of the set of all repelling periodic points, and applying the theory of 'holomomorphic motions' to this set.

Remark Notice that when we have a periodic attractor, once we have deformed the complex structure on the 'fundamental torus' A for the attractor, this determines the deformation everywhere in the basin of the attractor. If there were to exist a 'wandering component' of the Fatou set $F(q_c)$ for some c, we would have much more freedom to deform q_c: in fact (as Sullivan proved) we would have an infinite dimensional space of quadratic polynomials none of which would be conformally conjugate to another. This would contradict the fact that up to conformal conjugacy there exists only a one-complex-dimensional family of q_c's. (See Milnor, Appendix F, for the details of Sullivan's proof).

8.4 2nd Application: Bers simultaneous uniformisation: matings between Fuchsian groups

Recall that a *Fuchsian group* is a discrete subgroup of $PSL_2(\mathbb{R})$. Let G_1 be a geometrically finite Fuchsian group (a Fuchsian group which has a fundamental domain with a finite number of sides). Then G_1 acts (by fractional linear maps) on the upper half $\mathcal{U}$ of the complex plane. Suppose the limit set of this action is $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Of course G_1 also acts (by fractional linear maps) on the lower half plane $\mathcal{L}$ and the limit set of this action is also $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Let G_2 be another geometrically finite discrete subgroup of $PSL_2(\mathbb{R})$, such that G_2 is isomorphic to G_1 as an abstract group, and such that the action of G_2 on $\mathcal{U}$ is topologically conjugate to that of G_1 on $\mathcal{U}$.

Theorem 8.3 (Bers' Simultaneous Uniformisation Theorem)

Given subgroups G_1 and G_2 of $PSL_2(\mathbb{R})$ with the properties described above, there exists a discrete subgroup G of $PSL_2(\mathbb{C})$ the action of which on the Riemann sphere $\hat{\mathbb{C}}$ has the following properties:

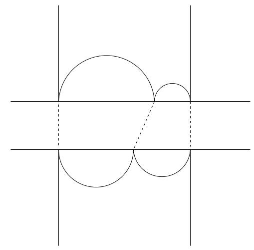

Figure 20: Fundamental domains for G_1 and G_2 on the upper and lower half-planes

(i) The limit set of the action is a quasicircle $\Lambda \subset \hat{\mathbb{C}}$.

(ii) On one component, U, of $\hat{\mathbb{C}} - \Lambda$ the action of G is conformally conjugate to the action of G_1 on $\mathcal{U}$.

(iii) On the other component, L, of $\hat{\mathbb{C}} - \Lambda$ the action of G is conformally conjugate to the action of G_2 on $\mathcal{L}$.

In the situation described by (i),(ii) and (iii) we might call the *Kleinian* group G (discrete subgroup of $PSL_2(\mathbb{C})$) acting on $\hat{C}$ a *mating* between the Fuchsian group G_1 (discrete subgroup of $PSL_2(\mathbb{R})$) acting on $\mathcal{U}$ and the Fuchsian group G_2 acting on $\mathcal{L}$. The action of G is a *holomorphic realisation* of the dynamical system obtained by gluing together the actions of G_1 on $\mathcal{U}$ and G_2 on $\mathcal{L}$ by means of a (topological) homeomorphism from the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ to the boundary $\partial\mathcal{L}$ of $\mathcal{L}$, which conjugates the action of G_1 on $\partial\mathcal{U}$ to that of G_2 on $\partial\mathcal{L}$.

Bers' Simultaneous Uniformization Theorem may be proved using the Measurable Riemann Mapping Theorem, as we now outline. Since G_1 is geometrically finite, the orbit space $\mathcal{L}/G_1$ is a Riemann surface with a finite number of marked cone points and puncture points. The orbit space $\mathcal{L}/G_2$ is a Riemann surface with a combinatorially identical set of data. It follows by standard Riemann surface theory that there exists a quasiconformal diffeomorphism $h : \mathcal{L}/G_1 \to \mathcal{L}/G_2$, sending marked points to marked points. The complex dilatation μ of h, when composed with the orbit projection, yields an L^∞ function $\mu : \mathcal{L} \to \mathbb{C}$, which we may extend to the whole of $\hat{\mathbb{C}}$ by defining $\mu(z)$ to be zero on $\hat{\mathbb{C}} - \mathcal{L} = \overline{\mathcal{U}}$. Equivalently, if one prefers to think in terms of measurable fields of ellipses, the field of ellipses defined by μ on $\mathcal{L}/G_1$ is pulled back to $\mathcal{L}$ and extended to the rest of $\hat{\mathbb{C}}$ by the standard (round) circle field on $\mathcal{U}$. By the measurable Riemann Mapping Theorem there now exists a quasiconformal diffeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ having complex dilatation μ. But, as is easily verified by the chain rule, each element of $G = \phi G_1 \phi^{-1}$ has complex dilatation zero, and so maps infinitesimal round circles to infinitesimal round circles. Thus G is a group of conformal automorphisms of $\hat{\mathbb{C}}$, that is to say a subgroup of $PSL_2(\mathbb{C})$. The limit set of G is the set $\Lambda = \phi(\hat{\mathbb{R}})$, which is a quasicircle by definition, since it is the image of a round circle under a quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Moreover ϕ provides a conformal conjugacy between the actions of G_1 on $\mathcal{U}$ and G on $U = \phi(\mathcal{U})$, and $\phi \circ h^{-1} : \mathcal{L} \to L = \phi(\mathcal{L})$ provides a conformal conjugacy between the actions of G_2 on $\mathcal{L}$ and G on $L = \phi(\mathcal{L})$, where here $h : \mathcal{L} \to \mathcal{L}$ denotes the lift of our quasiconformal diffeomorphism $h : \mathcal{L}/G_1 \to \mathcal{L}/G_2$.

A family of examples: once-punctured torus groups

Consider discrete representations of the free group F_2 on two generators X, Y in $PSL_2(\mathbb{R})$. Let A and B be elements of $PSL_2(\mathbb{R})$ representing X and Y, and restrict attention to the case that A and B are hyperbolic and their commutator $ABA^{-1}B^{-1}$ is parabolic. A generic representation of this kind has fundamental domain a quadrilateral in the upper half-plane, with all four vertices on the (completed) real line and all four sides geodesics in the hyperbolic metric, that is to say arcs of semicircles orthogonal to the real line: the group elements A and B identify pairs of opposite sides of this quadrilateral, and the orbit space is a punctured torus. The cross-ratio of the four vertices of a fundamental domain is a conjugacy invariant of the group (as a subgroup of $PSL_2(\mathbb{R})$). Any two representations G_1 and G_2 of this kind (which in general will have different have different cross-ratios) provide examples of groups to which we may apply Bers' Theorem. We consider G_1 acting on the upper half-plane and G_2 acting on the lower (Figure 20). In general their actions will only match combinatorially on their common limit set, the completed real axis, but if we glue them together combinatorially the conclusion of Bers' theorem tells us that we can realise this topological mating as a *holomorphic* dynamical system, a Kleinian group (discrete subgroup of $PSL_2(\mathbb{C})$) which has as its limit set a quasicircle in place of $\hat{\mathbb{R}}$.

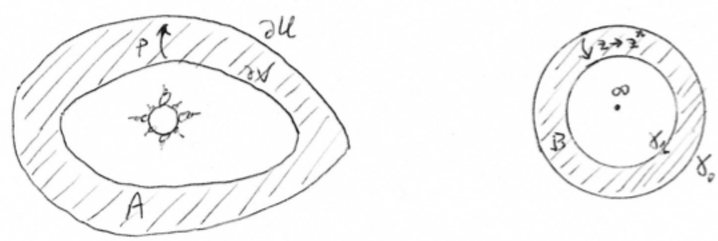

Figure 21: The Straightening Theorem

8.5 3rd Application: Polynomial-like mappings

In their paper *On the dynamics of polynomial-like mappings* (Ann Sci Ec Norm Sup 1985), Douady and Hubbard defined the notion of a *polynomial-like map.* This is a proper holomorphic surjection $p : V \to U$ where U and V are simply-connected open sets in $\mathbb{C}$ with $U \supset \overline{V}$. Such a map has a well-defined *filled Julia set* $K(p) = \bigcap_{n\geq 0} p^{-n}(\overline{V})$.

Theorem 8.4 (The Straightening Theorem of Douady and Hubbard) *For every polynomial-like mapping p there is a genuine polynomial map P which is hybrid equivalent to p.*

Here *hybrid equivalent* means that there is a quasiconformal conjugacy h between p and P on neighbourhoods of $K(p)$ and $K(P)$ such that the Beltrami form of h vanishes on $K(p)$ (so in particular h is conformal on $int(K(p))$. In the case that $K(p)$ is connected, we can think of this as a *mating* between the polynomial-like map p on its filled Julia set and the polynomial $z \to z^n$ on its filled Julia set.

At the heart of the proof of the Straightening Theorem is the Measurable Riemann Mapping Theorem. The idea is as follows. We suppose, for simplicity, that U and V are topological discs with smooth boundaries. A polynomial-like map p has a well-defined degree (the number of times p winds ∂V around ∂V, or equivalently the number of points in a generic $p^{-1}(z)$). Suppose this degree is n. $A = U - V$ is an annulus equipped with a map p of degree n from its inner boundary onto its outer boundary. Let B be the annulus between a circle γ_0 of radius $r > 1$ in $\mathbb{C}$ centred at the origin and its image γ_1 under $z \to z^n$. See Figure 21. It can be shown that it is possible construct a quasiconformal homeomorphism $h : A \to B$ conjugating the boundary map on A to the boundary map on B. Pull back the associated ellipse field on $U - V$ to an ellipse field on $U - K(p)$, by repeatedly applying p^{-1}, and extend it to the whole of U by using the standard field of round circles on $K(p)$. Now add the disc $D = \{z : |z| \geq r\} \cup \{\infty\}$ to U by pasting the annulus A to the annulus B using the quasiconformal homeomorphism h to do the pasting. The maps p on V and $z \to z^n$ on D fit neatly together to give a map a degree n map $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which preserves the ellipse field. Applying the Measurable Riemann Mapping Theorem to the ellipse field yields a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ straightening it to the field of round circles and now $fQf^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a genuine polynomial with the desired properties. QED

Tuning, renormalisation and baby Mandelbrot sets

It turns out that there are many parameter values c in the Mandelbrot set M where in some region of the dynamical plane the first return map $(q_c)^n$ is a quadratic-like map, which is then hybrid equivalent to some $q_{c'}$. This process of 'renormalisation' is the dynamical counterpart to the phenomenon of 'tuning' (replacing digits 0 and 1 in external ray addresses by finite strings of digits θ_- and θ_+), discussed briefly in Section 7; it follows from the Straightening Theorem that the Julia set of q_c then contains copies of $J(q_{c'})$. Families of quadratic-like first return maps satisfying certain conditions are called 'Mandelbrot-like families' by Douady and Hubbard in their paper (Ann Sci Ec Norm Sup 1985) : these give rise to 'baby Mandelbrot sets' - small copies of M on finer and finer scales. Indeed every point on the boundary of the Mandelbrot set M is an accumulation point

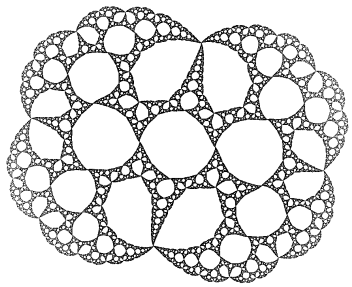

Figure 22: A quadratic rational map mating Douady's rabbit with $z \to z^2 - 1$

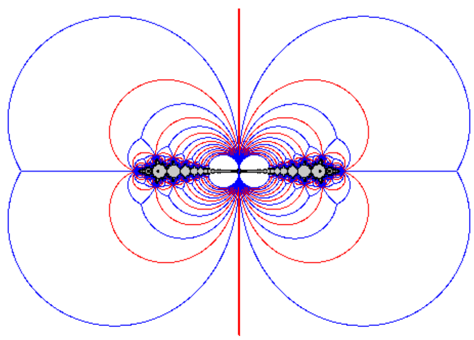

Figure 23: A holomorphic correspondence mating $z \to z^2 - 1$ with the modular group

of baby Mandelbrot sets (see McMullen's paper in the LMS Lecture Notes volume edited by Tan Lei).

Matings of polynomials

The construction described in the outline proof of Theorem 8.4 (the Straightening Theorem) can equally well be used to *mate* q_c with $q_{c'}$ for any pair c, c' in the interior of the cardioid M_0, that is to construct a *rational* map q of degree two which has Julia set $J(q)$ a quasicircle and which is conformally conjugate to q_c on one component of $\hat{\mathbb{C}} - J(q)$ and to $q_{c'}$ on the other component. A more challenging task is to mate q_c with $q_{c'}$ for c and c' elsewhere in the Mandelbrot set, as $J(q_c)$ and $J(q_{c'})$ are now *quotients* of the circle (in the case that they are locally connected) and of course in general they will be different quotients. Mary Rees and Tan Lei proved that *hyperbolic* quadratic polynomials $q_c, q_{c'}$ can be mated if and only if c' is not in the conjugate limb of M to that of c (the proof involves an application of Thurston's criterion for when a topological branched covering of the sphere is 'equivalent' to a rational map). See Figure 22. Finally we remark that if we extend our notions of rational maps and Kleinian groups to include *holomorphic correspondences*, it becomes possible to mate a hyperbolic quadratic polynomial with the modular group (Figure 23), but that is another story.... See the forthcoming book of Bodil Branner and Núria Fagella for the technical details of 'quasiconformal surgery' and many more applications.

The list of 'references' on the next page contains only books. Relevant journal articles have been referred to individually in the course of these Notes.

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