Holomorphic Dynamics and Hyperbolic Geometry (February-March 2013)

Assessment Exercises

These questions concerning Blaschke products are adapted from exercises in Milnor's book. They are not intended to be difficult. You can use any theorems from the course. You should send me your solutions - e.g. by e-mailing scanned scripts to me at s.r.bullett@qmul.ac.uk - before the end of March.

1. Show that for any $a \in \mathbb{D}$ the map:

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}$$

carries the unit circle to itself, and the origin to a point of $\mathbb{D}$, and hence carries the unit disc $\mathbb{D}$ isomorphically to itself. *[HINT: Observe that dividing the numerator and denominator of $\phi_a(e^{i\theta})$ by $e^{i\theta/2}$ gives an expression of the form $\zeta/\bar{\zeta}$.]*

2. A finite product of the form

$$(B) \qquad f(z) = e^{i\theta}\phi_{a_1}(z)\phi_{a_2}(z)\ldots\phi_{a_n}(z)$$

with $a_1, \ldots, a_n \in \mathbb{D}$, is called a *Blaschke product* of degree n.

Show that f is a rational map which carries $\mathbb{D}$ onto $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ onto $\hat{\mathbb{C}} \setminus \mathbb{D}$. Deduce that the unit circle S^1 is completely invariant and hence that the Julia set $J(f) \subseteq S^1$.

3. If $g(z) = 1/f(z)$, where f is a Blaschke product, show that $J(g)$ is also contained in the unit circle.

4. If f is a Blaschke product of the form (B) with $n \geq 2$ and one of its factors is $\phi_0(z) = z$, show that:
(i) f has an attracting fixed point at 0.
(ii) $1/f(1/z)$ is also a Blaschke product with one of its factors $\phi_0(z) = z$, so f has an attracting fixed point at ∞ as well as at 0.
(iii) Deduce that $J(f)$ is the entire circle. (You may assume without proof that for any attracting fixed point z_0 all points in the component of $F(f)$ containing z_0 have forward orbits which converge to z_0.)

5. Suppose that

$$f(z) = z\left(\frac{z-a}{1-az}\right)$$

with $a \in \mathbb{R}$ and $|a| < 1$ (so f satisfies the hypotheses of Question 4). Let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote the map $\psi(z) = z + 1/z$. Show that there is a unique rational map F such that $F\psi = \psi f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. In this way construct a 1-real-parameter family of non-conjugate quadratic rational maps with Julia set the real interval $[-2, +2]$, each with a fixed point at ∞. (You may assume that $J(F) = \psi(J(f))$, or you can prove this.)

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