Holomorphic Dynamics and Hyperbolic Geometry

Solutions 2

1. If f is a rational function with a fixed point at ∞ show that the multiplier λ at ∞ is equal to $\lim_{z\to\infty} 1/f'(z)$. Deduce that the fixed point at ∞ is a superattractor if and only if $\lim_{z\to\infty} f'(z) = \infty$. (Hint: consider the power series expansion around $\zeta = 0$ of $\sigma f \sigma$, where $\sigma(\zeta) = 1/\zeta$).

Solution. Let $g = \sigma f \sigma$ where $\sigma(z) = 1/z$. Then the multiplier of f at its fixed point ∞ is equal to the multiplier of g at its fixed point 0. But 0 is a superattractive fixed point of g if and only if the Taylor series for g around $z = 0$ has the form:

$$g(z) = a_k z^k + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \dots$$

for some $k \geq 2$ (with $a_k \neq 0$). So for large z, $f(z)$ has the form:

$$\begin{aligned} f(z) &= (a_k z^{-k} + a_{k+1} z^{-(k+1)} + a_{k+2} z^{-(k+2)} + \dots)^{-1} \\ &= z^k (a_k + a_{k+1} z^{-1} + a_{k+2} z^{-2} + \dots)^{-1} \\ &= a_k^{-1} z^k (1 + \dots)^{-1} = a_k^{-1} z^k (1 + \dots) \end{aligned}$$

where '...' is a power series in z^{-1}, so tends to zero as z tends to ∞. So $\lim_{z\to\infty} f'(z) = \lim_{z\to\infty} k a_k^{-1} z^{k-1}$ and as $k \geq 2$ we have $\lim_{z\to\infty} f'(z) = \infty$.

2. Picard's Theorem states that if a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ (i.e. an *entire* function) has the property that there are at least two points of $\mathbb{C}$ that are not in the image of f, then f is constant. Deduce Picard's Theorem from Liouville's Theorem and the fact that $\mathbb{D}$ is the universal cover of the thrice-punctured Riemann sphere $\hat{\mathbb{C}}$. Write down a non-constant entire function the image of which omits just one point of $\mathbb{C}$.

Solution. Suppose $f : \mathbb{C} \to (\hat{\mathbb{C}} - \{0, 1, \infty\})$ is holomorphic. Then since $\mathbb{C}$ is simply-connected, f lifts to a holomorphic function $\tilde{f} : \mathbb{C} \to \mathcal{H}_+$. Now if we let α be a Möbius transformation sending the half-plane $\mathcal{H}_+$ bijectively to the unit disc $\mathbb{D}$, the composite $\alpha \circ \tilde{f}$ is a bounded entire function, therefore constant (by Liouville's Theorem). Hence $\tilde{f}$ is constant. Hence f is constant.

3. Let f be a rational map. Using the 'normal families' definition of the Fatou set, prove that the Fatou set of f^2 (i.e. f composed with f) is the same set as the Fatou set $F(f)$ of F. Now consider $f(z) = z^2 - 1$. Show that $0, -1$ and ∞ are attracting fixed points of f^2 (i.e. f composed with f) and deduce that they are in different components of the Fatou set $F(f)$ of f. Deduce that $F(f)$ contains infinitely many components. Let F_0 denote the component containing 0. Sketch the position of the components of $f^{-n}(F_0)$ for $n = 1, 2, 3$, indicating how they map to each other under f.

Solution. $z \in F(f^2) \Rightarrow$ every infinite sequence in $\{f^{2n}\}_{n>0}$ has a subsequence which converges locally uniformly at z to a function g. Now any infinite sequence in $\{f^n\}_{n>0}$ either has a subsequence consisting of even powers, in which case there is a subsequence converging locally uniformly to g, or it has a subsequence consisting of odd powers, in which case there is a subsequence converging locally uniformly at z to $f \circ g$. Hence $f \in F(f)$.
Conversely, $f \in F(f) \Rightarrow$ the infinite family $\{f^{2n}\}_{n>0}$ has a subsequence which converges locally uniformly at z to a function g, and hence every infinite subfamily of $\{f^{2n}\}_{n>0}$ has a subsequence which converges locally uniformly at z to g, in other words $z \in F(f^2)$.

For $f(z) = z^2 - 1$ we have $f(f(z)) = (z^2-1)^2 - 1 = z^4 - 2z^2$. Writing $g(z)$ for $f(f(z))$, we have $g(0) = 0$, $g(-1) = -1$, $g'(0) = 0$ and $g'(-1) = 0$, so 0 and 1 are superattracting fixed points. Also ∞ is a superattracting fixed point since this is true for every polynomial of degree ≥ 2 (e.g. by the criterion in question 1). Every point in the component of an attacting fixed point has forward orbit converging to that fixed point, so 0, -1 and ∞ are in different components. (I'll draw a sketch in Week 4 to illustrate how the various components map to one another in this example.)

4. A non-identity element $\alpha \in PSL(2, \mathbb{R})$ is said to be:
elliptic if it has just one fixed point in the open upper half plane;
hyperbolic if it has two distinct fixed points on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$;
parabolic if it has just one fixed point in $\hat{\mathbb{C}}$ (necessarily on $\hat{\mathbb{R}}$).
(i) Regarding α as a 2×2 real matrix of determinant 1, show that α is elliptic, hyperbolic, parabolic $\Leftrightarrow |tr(\alpha)| < 2, > 2, = 2$ respectively (where the trace of a matrix is the sum of the entries on the main diagonal).
(ii) Show that if α is hyperbolic then it is conjugate in $PSL(2, \mathbb{R})$ to $z \to \lambda z$ for some non-zero $\lambda \in \mathbb{R}$, and in fact that we may require λ to be > 0.
(iii) Show that if α is parabolic then it is conjugate in $PSL(2, \mathbb{R})$ to $z \to z + 1$ or to $z \to z - 1$.
(iv) Show that in the Poincaré disc model of the hyperbolic plane the elliptic isometries fixing the origin are the Euclidean rotations.

Solution.
(i) The fixed points of $\alpha(z) = z$ in $\hat{\mathbb{C}}$ are the solutions of $z(cz+d) = az+b$, i.e.

$$cz^2 + (d-a)z - b = 0$$

(where if $c = 0$ then one of the fixed points is ∞, and if $c = 0$ and $d = a$ then ∞ is the only fixed point).
When $c \neq 0$ we get (since a, b, c, d are real and $ad - bc = 1$):
• one solution, necessarily real, if $(d-a)^2 + 4bc = 0$ i.e. if $(d+a)^2 = 4$;
• two distinct real solutions if $(d-a)^2 + 4bc > 0$ i.e. if $(d+a)^2 > 4$;
• one solution in the upper half plane and another (the complex conjugate) in the lower half plane if $(d-a)^2 + 4bc < 0$ i.e. if $(d+a)^2 < 4$.

(ii) If α is hyperbolic we can move the fixed points to 0 and ∞ by a (real) Möbius conjugacy. Now α has the form $z \to \lambda z$ for some real λ, and as α maps the uper half-plane to itself we have $\lambda > 0$.
(In fact we may choose $\lambda > 1$, which is what I intended to ask, for if $\lambda < 1$ then by exchanging 0 and ∞ we can conjugate α to $z \to \lambda^{-1}z$.)

(iii) If α is parabolic then by a Möbius conjugacy we can assume the unique fixed point is at ∞. Then α has the form $z \to z + \lambda$ for some $0 \neq \lambda \in \mathbb{R}$. Now if we conjugate such an α by $z \to \mu z$ it becomes $z \to \mu(\mu^{-1}z + \lambda)$. So by taking $\mu = 1/|\lambda|$ we can conjugate α either $z \to z+1$ or to $z \to z-1$.

(iv) In the Poincaré disc model the isometries have the form

$$z \to e^{i\theta}\frac{z-a}{1-\bar{a}z} \text{ with } a \in \mathbb{D}.$$

Any isometry fixing 0 has $a = 0$ and so is of the form $z \to e^{i\theta}z$.

5. On the hyperbolic plane a *reflection* is an orientation-reversing isometry β which fixes some geodesic pointwise.
(i) Show that every reflection β is an *involution* (i.e. $\beta^2 = I$);
(ii) Show that for every reflection β there is an element of $PSL(2,\mathbb{R})$ which conjugates β to 'reflection in the imaginary axis', i.e. the map $z \to -\bar{z}$.
(iii) Show that every orientation-preserving isometry of the hyperbolic plane can be written as the composition of a pair of reflections (by the previous question it will suffice to consider $z \to \lambda z$ and $z \to z \pm 1$ on $\mathcal{H}_+$, and $z \to e^{i\theta}z$ on $\mathbb{D}$). Deduce that every orientation-reversing isometry can be written as a composition of at most three reflections.
(iv) Show that the orientation-reversing isometries of the hyperbolic plane are precisely the maps

$$z \to \frac{a\bar{z}+b}{c\bar{z}+d} \qquad a, b, c, d \in \mathbb{R},\ ad - bc = -1$$

(Hint: if an isometry reverses orientation then composing it with a reflection will preserve orientation.)

Solution.

(i) If β fixes the geodesic γ pointwise, then for every $p \in \gamma$ the geodesic γ' through p orthogonal to γ is sent to itself. Hence β^2 is an orientation-preserving isometry of γ'. So β^2 is the identity on the end points of γ' as well as on the end-points of γ. Hence $\beta^2 = I$ (as a Möbius transformation with ≥ 3 fixed points is the identity).

(ii) Conjugate the fixed geodesic of β to the imaginary axis in the upper half plane. Now by the argument in the solution to (i) above β sends each semicircle orthogonal to this axis to itself, fixing the intersection point of the semicircle with the imaginary axis and preserving (hyperbolic) distances. So β is the Euclidean reflection in this axis ($z \to -\bar{z}$).

(iii) $z \to z+1$ is reflection in the imaginary axis followed by reflection in the vertical line $Re(z) = 1/2$. $z \to z-1$ is the same pair of reflections in the opposite order.

$z \to \lambda z$ is reflection in the semicircle through i orthogonal to the imaginary axis, followed by reflection in the semicircle through $i\sqrt{\lambda}$ orthogonal to the imaginary axis.

Rotation through θ about the origin, in the disc model, is composition of reflections in lines through the origin at angle $\theta/2$ to one another.

if α is an orientation-reversing isometry, then $\beta\alpha$ preserves orientation, where $\beta(z) = -\bar{z}$, so by the argument above $\beta\alpha = R_2R_1$ (a product of two reflections. Hence $\alpha = \beta R_2 R_1$ (since $\beta^2 = I$).

(iv) If α reverses orientation then $\alpha\beta$ preserves orientation, where $\beta(z) = -\bar{z}$. Hence

$$\alpha\beta(z) = \frac{az+b}{cz+d} \text{ with } a,b,c,d \in \mathbb{R}, ab-cd=1$$

So

$$\alpha(z) = \frac{a(-\bar{z})+b}{c(-\bar{z})+d} \text{ with } a,b,c,d \in \mathbb{R}, ab-cd=1$$

$$= \frac{(-a)\bar{z}+b}{(-c)\bar{z}+d} \text{ with } -a,b,-c,d \in \mathbb{R}, (-a)b-(-c)d=-1.$$