MAS115 Calculus I 2006-2007

Problem sheet for exercise class 1

- **Make sure you attend the excercise class that you have been assigned to!**
- The instructor will present the starred problems in class.
- You should then work on the other problems on your own.
- The instructor and helper will be available for questions.
- Solutions will be available online by Friday.

(*) Problem 1: Prove that for all positive real numbers x and y (i.e. $x, y \in \mathbb{R}^+$),

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy}$$

(a) by direct proof, and
(b) by using the geometric-arithmetic inequality.

(*) Problem 2: Determine the set of all real numbers x (i.e. $x \in \mathbb{R}$) that satisfy

$$|2x - 1| + |4x + 1| < 3$$

(a) by direct computation, and
(b) by plotting the graph.

Problem 3: Determine the set of all real numbers x (i.e. $x \in \mathbb{R}$) that satisfy

$$x^2 - 3x - 4 < 0$$

(a) by direct computation, and
(b) by plotting the graph of $y = x^2 - 3x - 4$.
Hint: compute the zeros of $x^2 - 3x - 4$.

Problem 4: Determine the set of all real numbers x (i.e. $x \in \mathbb{R}$) that satisfy

$$\sqrt{1 - x^2} \leq -x$$

(a) by direct computation, and
(b) by plotting the graphs of $y = -x$ and $y = \sqrt{1 - x^2}$.

Extra: Prove that for all real numbers x and y (i.e. $x, y \in \mathbb{R}$)

$$\big||x| - |y|\big| = |x + y| + |x - y| - |x| - |y| \, .$$

Thomas Prellberg, September 2006

<u>Problem 1</u> You have seen the arithmetic-geometric inequality in class, including a proof.

(a) Direct calculation

$$(*) \quad \frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \quad \text{for } x,y>0$$

$$\Leftrightarrow \frac{2xy}{x+y} \leq \sqrt{xy} \quad \Big|\,(\;)^2 \quad [\text{why} \Leftrightarrow ?]$$

$$\Leftrightarrow \frac{4(xy)^2}{(x+y)^2} \leq xy \quad \Big|\times \frac{(x+y)^2}{xy} \quad [\text{why} \Leftrightarrow ?]$$

$$\Leftrightarrow 4xy \leq (x+y)^2 \quad \big|-4xy$$

$$\Leftrightarrow 0 \leq (x+y)^2 - 4xy$$

$$\Leftrightarrow 0 \leq x^2+2xy+y^2-4xy$$

$$= x^2-2xy+y^2 = (x-y)^2$$

Thus (*) is equivalent to

$$0 \leq (x-y)^2$$

which is true for all $x, y \geq 0$

(b) Using $\sqrt{xy} \leq \frac{1}{2}(x+y)$

Rewrite $\dfrac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \quad \Big| \cdot \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)$

$$\Leftrightarrow \quad 1 \leq \sqrt{xy}\,\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right) \quad \Big| \cdot \frac{1}{\sqrt{xy}}$$

$$\Leftrightarrow \quad \frac{1}{\sqrt{xy}} \leq \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)$$

$$\Leftrightarrow \quad \sqrt{\frac{1}{x}}\sqrt{\frac{1}{y}} \leq \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)$$

this is $\sqrt{xy} \leq \frac{1}{2}(x+y)$ with x, y replaced by $\frac{1}{x}$ and $\frac{1}{y}$

Problem 2

$$|2x-1| + |4x+1| < 3$$

(a) $2x-1<0$ and $4x+1<0$

$\iff$ $x<\frac{1}{2}$ and $x<-\frac{1}{4}$

i.e. $\underline{\underline{x \in (-\infty, -\frac{1}{4})}}$

Then $|2x-1| + |4x+1|$

$= -(2x-1) - (4x+1) = -6x$

$-6x < 3 \iff x > -\frac{1}{2}$

so $x \in (-\infty, -\frac{1}{4}) \cap (-\frac{1}{2}, \infty) = \underline{\underline{(-\frac{1}{2}, -\frac{1}{4}]}}$

(b) $2x-1<0$ and $4x+1 \geq 0$

$\iff$ $x<\frac{1}{2}$ and $x \geq -\frac{1}{4}$

i.e. $\underline{\underline{x \in [-\frac{1}{4}, \frac{1}{2})}}$

Then $|2x-1| + |4x+1|$

$$= -(2x-1) + (4x+1) = 2x+2$$

$$2x+2 \leq 3 \iff x \leq \frac{1}{2}$$

so $x \in [-\frac{1}{4}, \frac{1}{2}) \cap (-\infty, \frac{1}{2}) = \underline{\underline{[-\frac{1}{4}, \frac{1}{2})}}$

(c) $\quad 2x-1 \geq 0$ and $4x+1 < 0$

$\iff \quad x \geq \frac{1}{2}$ and $x < -\frac{1}{4}$

this is not possible for any $x \in \mathbb{R}$; $\underline{\underline{x \in \emptyset}}$

(d) $\quad 2x-1 \geq 0$ and $4x+1 \geq 0$

$\iff \quad x \geq \frac{1}{2}$ and $x \geq -\frac{1}{4}$

i.e. $x \in [\frac{1}{2}, \infty)$

Then $|2x-1| + |4x+1|$

$$= (2x-1) + (4x+1) = 6x$$

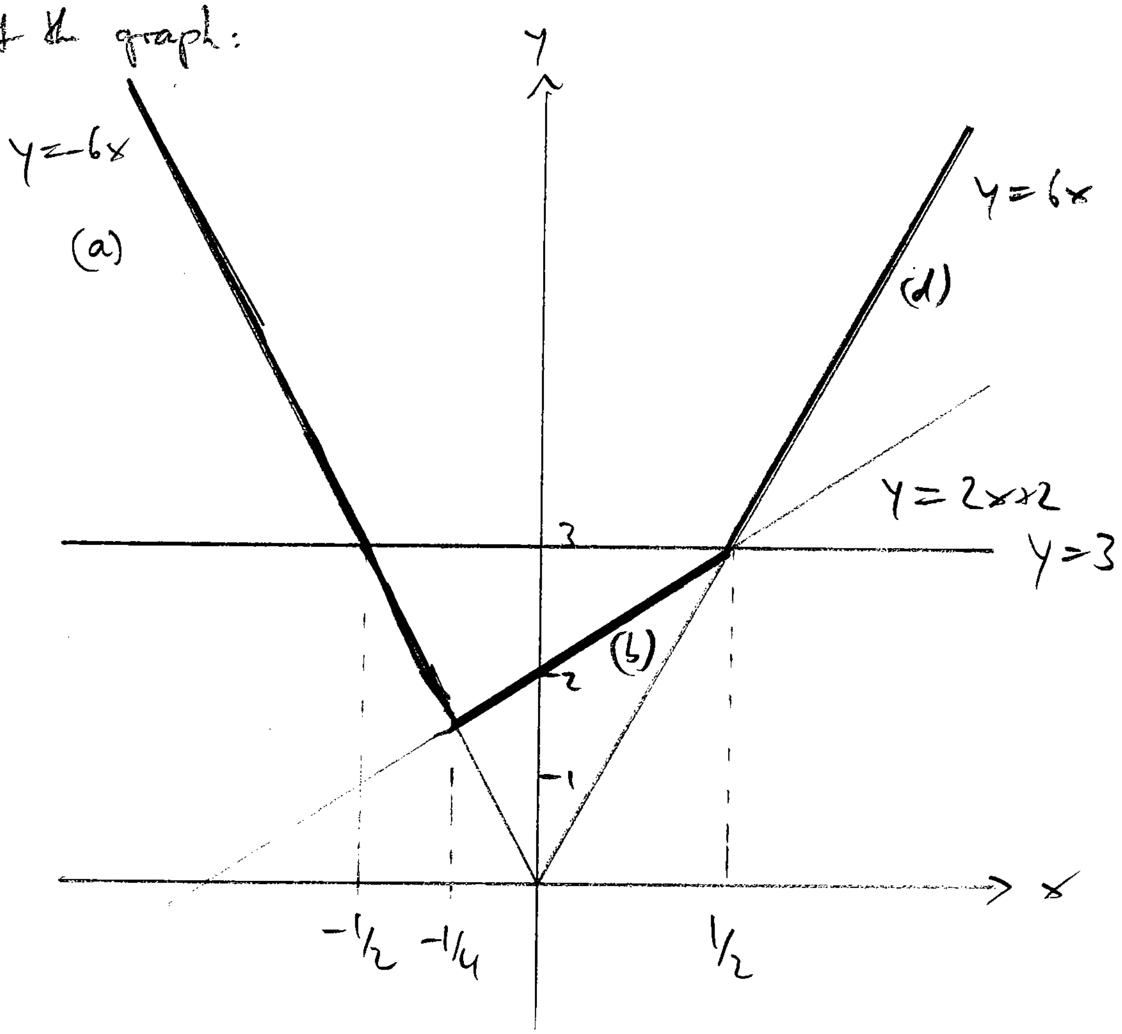
$$6x < 3 \iff x < \tfrac{1}{2}$$

so $x \in [\tfrac{1}{2}, \infty) \cap (-\infty, \tfrac{1}{2}) = \underline{\underline{\emptyset}}$

Together,

$$\underline{\underline{x}} \in (-\tfrac{1}{2}, -\tfrac{1}{4}] \cup (-\tfrac{1}{4}, \tfrac{1}{2}) = (-\tfrac{1}{2}, \tfrac{1}{2})$$

Plot the graph:



Problem 3 $\quad x^2 - 3x - 4 < 0$

Hint: $\quad x^2 - 3x - 4 = 0$

$$\Leftrightarrow x_{1,2} = \frac{3 \mp \sqrt{9+16}}{2}$$

$$x_1 = -1, \quad x_2 = 4$$

$$x^2 - 3x - 4 = (x - x_1)(x - x_2)$$

$$= (x+1)(x-4)$$

$$x^2 - 3x - 4 < 0 \Leftrightarrow (x+1)(x-4) < 0$$

Therefore (a) $x + 1 > 0$ and $x - 4 < 0$

or (b) $x + 1 < 0$ and $x - 4 > 0$

(a) gives $x > -1$ and $x < 4 \Rightarrow x \in (-1, 4)$

(b) gives $x < -1$ and $x > 4 \Rightarrow x \in \emptyset$

Solution: $\underline{\underline{x \in (-1, 4)}}$

4) $$\sqrt{1-x^2} \le -x$$

Observe

- lhs is only defined for $x \in [-1,1]$
- lhs is non-negative, therefore $x \le 0$

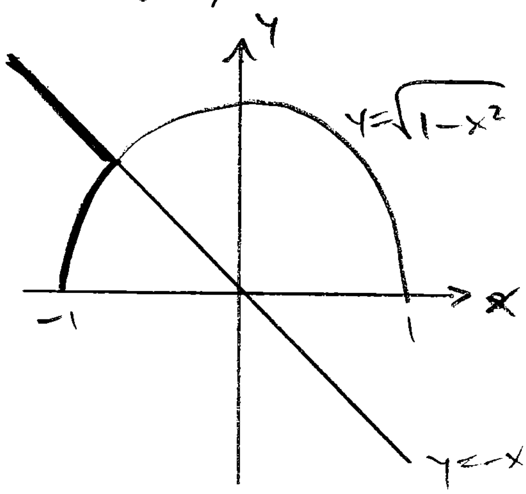
squaring both sides gives

$$1-x^2 \le x^2$$

$$\iff \frac{1}{2} \le x^2$$

$$\iff x \le -\frac{\sqrt{2}}{2} \quad \text{or} \quad x \ge +\frac{\sqrt{2}}{2}$$

but $x \in [-1,0]$, so $x \in \left[-1, -\frac{\sqrt{2}}{2}\right]$



<u>Extra</u> lots of different cases

$$x \geq y \geq 0, \quad x \geq 0 \geq y, \quad 0 \geq x \geq y$$

$$x \leq y \leq 0, \quad x \leq 0 \leq y, \quad 0 \leq x \leq y$$

e.g. for $x \geq 0 \geq y$

the lhs gives

$$||x| - |y|| = |x + y|$$

and the rhs gives

$$|x+y| + |x-y| - |x| - |y|$$

$$= |x+y| + x - y - x + y = |x+y|$$

which are equal.